Robust Repositioning for Vehicle Sharing

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Problem definition: In this paper, we study the fleet repositioning problem for a free-float vehicle sharing system, aiming to dynamically match the vehicle supply and travel demand at the lowest total cost of repositioning and lost sales.

Academic/Practical relevance: Besides the analytical results on the optimal repositioning policy, the proposed optimization framework is applicable to practical problems by its computational efficiency as well as the capability to handle temporally dependent demands.

Methodology: We first formulate the problem as a stochastic dynamic program. To solve for a multi-region system, we deploy the distributionally robust optimization (DRO) approach that can incorporate demand temporal dependence, motivated by real data. We first propose a "myopic" two-stage DRO model that serves both as an illustration of the DRO framework as well as a benchmark for the later multi-stage model. We then develop a computationally efficient multi-stage DRO model with enhanced linear decision rule (ELDR).

Results: Under a 2-region system, we find that a simple reposition up-to and down-to policy to be optimal, when the demands are temporally independent. Such structure is also preserved by our ELDR solution. We also provide new analytical insights by proving the optimality of ELDR in solving single-period DRO problem. We then show that the numerical performance of ELDR solution is close to the exact optimal solution from the dynamic program.

Managerial implications: In a real-world case study of car2go, we quantify the "value of repositioning" and compare with several benchmarks to demonstrate that the ELDR solutions are computationally scalable and in general result in lower cost with less frequent repositioning. We also explore several managerial implications and extensions from the experiments.

Key words: fleet repositioning; vehicle sharing; dynamic program; robust optimization.
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1. Introduction

Bike sharing systems are not new, since almost every major city from New York (e.g, Citi Bike) to London (e.g., Santander Cycles) to Paris (e.g., Vélib') has one. Unlike the station-based systems in those cities where bikes are kept at docks, the free-float bike sharing systems allow them to be picked up and left anywhere in well-defined service regions, saving the last-mile walking from nearby bike stations to final destinations for riders. Thanks

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to the mobile applications for unlocking and tracking bikes, various bike sharing systems, especially free-float ones such as Mobike and ofo, have brought more than 2 million new bikes to Chinese city streets (Financial Times 2017). While the free-float systems create convenience for riders, they also frustrate city authorities. On April 4, 2017, it was reported that 10,000 bicycles entered the Shenzhen Bay Park (Shenzhen, China) making tourists nearly unable to walk along the bay. To prevent similar instances in the future, the city authorities of Shenzhen issued a ban on bikes inside the park for several days (Forbes 2017).

Such fleet management problem is not unique to bike sharing. It also occurs in car sharing systems and is particularly critical when the systems are free-float. In the early station-based vehicle sharing systems, such as Zipcar, customers are usually required to return the vehicles to the same pickup stations and hence with the right number of vehicles at each station, little repositioning is needed. To accommodate one-way trips, free-float systems utilize the on-street parking spaces and allow customers to return the vehicles to any parking spaces within its service region. The added flexibility makes fleet operations more difficult. Compared to bikes, which can be repositioned in batches via trucks, repositioning of cars certainly does not enjoy such economy of scale. Indeed, it can be quite costly to reposition cars in cities. For example, car2go in Brooklyn dispatches a 40-person squad to patrol "the borough for vehicles that are out of gas, illegally parked, too densely packed in one region, or otherwise causing problems" and despite the efforts "you can't always find a car when you need one"(New York Magazine 2015).

While fleet repositioning is a critical operational decision, it is also important to the strategic and tactical decisions in vehicle sharing business. When designing the service region and planning the fleet configuration for a free-float vehicle sharing system, the firm needs to consider the long-run operating cost in fleet management in the optimization of strategic decisions (e.g., He et al. 2017, Lu et al. 2017). The modeling of fleet repositioning is usually simplified in the models developed for the strategic planning. In this paper, we focus on the operational problem of fleet repositioning for a free-float vehicle sharing system, to meet random demands at the right places and right time.

We model the system as a network and first formulate the fleet repositioning problem as a stochastic dynamic program in Section 3. The optimality of reposition up-to and down-to policy is established for a 2-region system, under temporal independence of demand.

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The general stochastic dynamic program, however, is computationally intractable due to "curse of dimensionality", and it requires full distribution information of a high dimensional random vector. To address both the issues of computational tractability and limited information, in Section 4, we propose approximate solutions based on the distributionally robust optimization (DRO) framework for multi-region systems with demand temporal dependence. We first discuss a single period two-stage problem as an illustration of the DRO framework. The exact solution to the single period problem—which we refer to as the "myopic" solution—also serves as a benchmark for the later proposed heuristic for multi-period problem. We then consider the general multi-period problem, for which we develop a computationally efficient approximation algorithm using the enhanced linear decision rule. Interestingly, we find that the solution from the approximation algorithm is exact for the single period robust problem, and it preserves the reposition up-to and down-to structure for a 2-region system. We examine numerically the performance of the proposed solutions in Section 5. For a 2-region system, the multi-period robust solution performs closely to the optimal solution from solving the dynamic program, which becomes computationally intractable when the number of regions becomes large. Using a set of real-world car sharing operational data, we quantify the value of repositioning and demonstrate that the multi-period robust solution is still computationally efficient for multi-region system. We also explore the impact of fleet size on the value of repositioning, and compare the repositioning patterns to the demand patterns from the numerical experiments. Finally, we extend the proposed ELDR approach to solve for the cases with spatial-temporal correlations information and repositioning capacity constraint, and discuss practical considerations in its implementation.

2. Literature Review

The emerging sharing economy has encouraged the innovations in shared mobility systems, including peer-to-peer sharing (e.g., Turo), ride sharing (e.g., Uber and Didi) and vehicle sharing (e.g., car2go and Mobike). In the operations management literature, several papers have studied the former two systems. In the context of peer-to-peer marketplaces, Jiang and Tian (2016) and Fraiberger and Sundararajan (2015) study customer's decision of purchases or rental while Benjaafar et al. (2018) examine the impact of sharing on ownership and usage under factors such as rental price, cost of ownership and moral hazard cost, etc.

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In the context of on-demand platforms with self-scheduled service providers (such as ride sharing platforms), Bimpikis et al. (2016), Tang et al. (2016), Cachon et al. (2017) and Taylor (2017) all consider using price and wage as a measure to mitigate the mismatch between demand and supply even though they differ significantly in modeling behaviors of customers as well as service providers. Hu and Zhou (2016), on the other hand, studies the more operational decisions of matching heterogeneous supply and demand with price and wage exogenously given. In comparison, our paper studies the third system of vehicle sharing in which the firm (rather than peers or self-scheduled service providers) primarily controls the fleet. While we also consider an operational level problem with exogenously given prices, our problem differs from that in Hu and Zhou (2016) in the sense that the firm is unable to ration the demands by destinations, as opposed to the ability of prioritizing certain type of demands in Hu and Zhou (2016), since in a free-float system, the firm has no prior information about a particular customer's destination.

Recently, there are also a number of papers that study various vehicle sharing systems. Bellos et al. (2017) discuss the manufacturer's strategy in offering car sharing business. Since their model considers the station-based system that accommodates only round trips, the fleet operations can be decoupled by each station and thus repositioning is not needed. To estimate the effects of service levels on ridership, Kabra et al. (2016) measure the accessibility of stations and the availability of bicycles using a set of operational data from a bicycle sharing system in Paris. He et al. (2017) optimize the service region for free-float electric vehicle sharing systems by modeling both the customer adoption behavior in response to the service coverage and the fleet operations, e.g., repositioning and recharging, using a queueing network. It is recognized in the above mentioned papers that vehicle availability is crucial to maintain a desirable service level, but none of them have explicitly discussed how to dynamically reposition vehicles among different regions.

The problem of repositioning a vehicle, or more generally an item, from one place to another has long been investigated in both the operations research and transportation literature. In the context of inventory management, the repositioning of inventories is also known as the transshipment. We refer the readers to Paterson et al. (2011) for a comprehensive survey. While fleet repositioning share some similarities as inventory transshipment, it differs in the key aspect that the replenishment of inventories (vehicles) can (and only) come from both repositioning as well as "consumption" of inventory at another location.

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This feature, in the two-location case, actually simplifies the analysis compared to the inventory transshipment problem. In addition, it is pointed out in Paterson et al. (2011) that generally very limited results are available for optimal policies in a multi-location problem and "[t]his leaves the possibility that current research on multi-location problem could be developed further using different ideas so that better transshipment policies can be found" (p. 134). To deal with the multi-region repositioning problem, in this paper we propose a distributionally robust approach in finding good repositioning policies and our numerical studies demonstrate that our approach is both computationally efficient and achieves a satisfactory performance comparing to several benchmarks.

There are several related transportation problems, that have been studied in the past decades. Shu et al. (2013) study the bicycle redistribution problem for bike sharing systems using a spatial-temporal network flow model. Under a similar stochastic model, Nair and Miller-Hooks (2011) develop a mixed-integer program with joint chance constraints. With trip data from New York's Citi Bike sharing, O'Mahony and Shmoys (2015) estimate the demand flows and formulate a mixed-integer program for overnight repositioning. The fleet repositioning problem has also been studied in Kek et al. (2009), Febbraro et al. (2012), Boyacı et al. (2015) and Nourinejad et al. (2015) for one-way station-based systems. However, in the above optimization models for fleet repositioning, the demands are assumed to be deterministic or follow Poisson processes. Built upon spatial-temporal networks, Erera et al. (2009) and Lu et al. (2017) consider the future demand uncertainty in the repositioning problems. Using an uncertainty budget concept from Bertsimas and Sim (2003), Erera et al. (2009) propose robust dynamic empty container repositioning model as a mixed integer program. Lu et al. (2017) employ a two-stage stochastic integer programming model, where the fleet and parking lots investment decisions are made in the first stage and fleet operations are modeled as recourse decisions in the second stage. In our study, we first characterize the structure of the optimal policy in the case of stochastic dynamic program and then develop a practical approximation algorithm for the problem under general time-varying random demands with possibly only partial distributional information.

A recent paper by Benjaafar et al. (2017) is most closely related to our work. Benjaafar et al. (2017) also study the structural properties of the optimal repositioning policy in a product rental network setting. Specifically, they show that the optimal policy is not to reposition anything when the state lies in a certain region and to reposition to the

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boundary of the region when the state is outside of the region. When the rental network consists of only two locations, their policy reduces to our reposition up-to and down-to policy. While Benjaafar et al. (2017) provide a more general characterization than ours, we point out some key differences here. In terms of proof, Benjaafar et al. (2017) establish the convexity of the value function and the structure of the policy via an intricate analysis of the derivatives of the value function. Our proof of the convexity, on the other hand, is based on a simple equivalent formulation commonly used in lost-sales inventory models. Our characterization of policy—although restricted to the 2-region case—is also based on a simple submodular (and derivative free) argument and can be used to characterize the structure of our robust repositioning policy as well. At a higher level, the focus of Benjaafar et al. (2017) is mainly on a theoretical characterization of the optimal policy. Our emphasis, however, is on a computationally efficient solution approach, which we complement with some structural understanding and a real-world case study.

3. Fleet Repositioning Problem

We consider the fleet repositioning problem in a free-float vehicle sharing system that operates in a well-defined service region. Different from the conventional station-based systems, one-way trips are allowed in the free-float system where customers can pickup any nearby available vehicles. Without informing the system about their destinations, the customers are only required to return the vehicles anywhere in the service region to end their trips.

Suppose the firm provides service to N regions as a network and customers can travel between any two regions in the network. We use $[N]$ to denote the set of running indices, i.e., $\{1,\ldots,N\}$. We let w_{ij} be the number of one-way trips by customers from region i to region j, w_{ii} be the number of round trips by customers that start and end in region i, and r_{ij} be the number of repositioning trips by the firm. In practice, the firm conducts repositioning regularly over T periods a day. When the firm conducts repositioning in 4 shifts a day, we have $T=4$. Since T is usually not large, we assume that both the customer trips and repositioning trips can be completed within a period. For instance, if $T=4$, the average period of a shift is 6 hours that is sufficient to complete a trip within a city. Therefore, we omit the travel time in our model for the ease of exposition.

To formally set up the mathematical model, in the following we refer bold faced characters such as $\mathbf{x}\in\mathbb{R}^N$ and $\boldsymbol{A}\in\mathbb{R}^{M\times N}$ to vectors and matrices, and x_i to the ith element of

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the vector $\mathbf{x}$. The special vectors $\mathbf{e}_i$ and $\mathbf{1}$ are dedicated to the unit vector with 1 at the ith element and the vector of all ones respectively.

We model the fleet repositioning problem as a stochastic dynamic program with planning horizon of T periods. At each period t, the outbound travel demand from region i is random and is denoted as d_{it}. We further denote $\mathbf{d_t} = (d_{it})$ to be the vector of demands at all regions in period t and $\mathbf{d_{[t]}} = (\mathbf{d_1}, \ldots, \mathbf{d_t})$ to be the historical demand realizations up to period t. The demands at all regions during the whole planning horizon $\mathbf{d_{[T]}}$ is assumed to follow a known joint probability distribution $\mathbb{P}$. At the beginning of period t, the firm observes the physical distribution of the fleet $\mathbf{x_t} = (x_{it})$ with x_{it} vehicles in region i and the historical demand realizations $\mathbf{d_{[t-1]}}$ as the system state. Before any customer arrivals, the firm makes repositioning decisions $\mathbf{r_t} = (r_{ijt})$ where r_{ijt} vehicles are repositioned from i to j at the cost $s_{ijt} > 0$ per trip. After the repositioning, outbound travel demand d_{it} arrives at region i and picks up available vehicles. We assume that each vehicle is used to satisfy at most one trip in each period (also seen in Benjaafar et al. (2017)). A customer arrival finding no vehicles available at i is lost with a penalty $p_{ijt} > 0$ if her intended destination is j. Note that in a free-float system, the firm is usually not able to ration the demands based on their destinations. We assume, however, that the firm has the knowledge over α_{ijt}, the probability that a customer trip originating from i at period t that ends at j. Therefore, by letting w_{it} be the total fulfilled customer trips from i and w_{ijt} be the fulfilled customer trips from i to j, we can write $w_{ijt} = \alpha_{ijt} w_{it}$ with $\sum_{j\in[N]} \alpha_{ijt} = 1$. We then define the average lost sales penalty of a customer trip from region i as $\bar{p}_{it} = \sum_{j\in[N]} \alpha_{ijt} p_{ijt}$. We formulate the following stochastic dynamic program (DP) in (1) to minimize the expected total repositioning cost and lost sales penalty:

$$V_t\left(\mathbf{x_t}, \mathbf{d_{[t-1]}}\right) = \min_{\substack{\mathbf{r_t} \geq 0 \\ 0 \leq \sum_{j\in[N]} r_{ijt} \leq x_{it}}} \left\{ \sum_{i,j\in[N]} s_{ijt} r_{ijt} + \mathbb{E}_{\mathbb{P}}\left[J_t\left(\mathbf{x_t}, \mathbf{r_t}, \mathbf{d_{[t]}}\right)\right] \right\}. \tag{1}$$

In (1), the expectation is taken over the conditional probability of $\mathbf{d_t}$ given $\mathbf{d_{[t-1]}}$, and

$$J_t\left(\mathbf{x_t}, \mathbf{r_t}, \mathbf{d_{[t]}}\right) = \sum_{i\in[N]} \bar{p}_{it}\left(d_{it} - w_{it}\right) + V_{t+1}\left(\mathbf{x_{t+1}}, \mathbf{d_{[t]}}\right),$$

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where

$$x_{i(t+1)} = x_{it} + \sum_{j\in[N]} r_{jit} - \sum_{j\in[N]} r_{ijt} + \sum_{j\in[N]} \alpha_{jit} w_{jt} - w_{it},\ \forall i \in [N], t \in [T]$$

$$w_{it} = \min\left\{ d_{it}, x_{it} + \sum_{j\in[N]} r_{jit} - \sum_{j\in[N]} r_{ijt} \right\},\ \forall i \in [N], t \in [T].$$

and the terminal cost $V_{T+1}(\mathbf{x_{T+1}}) = 0$.

The constraint $\sum_{j\in[N]} r_{ijt} \le x_{it}$ in (1) ensures that the total repositioning departures from i: $\sum_{j\in[N]} r_{ijt}$ do not exceed the available vehicles x_{it} *before repositioning*. In the expression of $J_t(\mathbf{x_t}, \mathbf{r_t}, \mathbf{d_{[t]}})$, the number of available vehicles $x_{i(t+1)}$ for the next period is the sum of the number of available vehicles *after repositioning*: $x_{it} + \sum_{j\in[N]} r_{jit} - \sum_{j\in[N]} r_{ijt}$ and the net inflow of vehicles from the fulfilled demands: $\sum_{j\in[N]} \alpha_{jit} w_{jt} - w_{it}$. The fulfilled demand w_{it} at region i is either the total demand: d_{it} or the number of available vehicles after repositioning: $x_{it} + \sum_{j\in[N]} r_{jit} - \sum_{j\in[N]} r_{ijt}$, whichever is smaller.

To facilitate the discussion, we use $\wedge$ and $\vee$ as minimum and maximum operators, i.e., $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$ for real numbers a and b. In the following, we show that under a certain condition on the input cost parameters p_{ijt}, s_{ijt}, the value function $V_t(\mathbf{x_t}, \mathbf{d_{[t-1]}})$ is convex in $\mathbf{x_t}$ for any $t \in [T]$.

LEMMA 1. *Suppose $\bar{p}_{it} \ge \sum_{j\ne i} s_{ji(t+1)} \alpha_{ijt}$ for any $i \in [N]$ and $t \in [T]$. Then,*

$$\begin{aligned} J_t\left(\mathbf{x_t}, \mathbf{r_t}, \mathbf{d_{[t]}}\right) = \min_{\mathbf{w_t}} & \left\{ \sum_{i\in[N]} \bar{p}_{it}\left(d_{it} - w_{it}\right) + V_{t+1}\left(\mathbf{x_{t+1}}, \mathbf{d_{[t]}}\right) \right\}, \\ \text{s.t.} \quad x_{i(t+1)} &= x_{it} + \sum_{j\in[N]} r_{jit} - \sum_{j\in[N]} r_{ijt} + \sum_{j\in[N]} \alpha_{jit} w_{jt} - w_{it},\ \forall i \in [N], \\ w_{it} &\le d_{it} \wedge \left(x_{it} + \sum_{j\in[N]} r_{jit} - \sum_{j\in[N]} r_{ijt} \right),\ \forall i \in [N], \end{aligned} \tag{2}$$

and $V_t\left(\mathbf{x_t}, \mathbf{d_{[t-1]}}\right)$ is convex in $\mathbf{x_t}$ for any $t \in [T]$ and demand realization $\mathbf{d_{[t-1]}}$.

Proof: Please see Appendix A.1 in the Online Supplement.

The condition $\bar{p}_{it} \ge \sum_{j\ne i} s_{ji(t+1)} \alpha_{ijt}$ in Lemma 1 (a similar condition is also imposed in Benjaafar et al. (2017)) simply says that the average profit of satisfying a trip departing from i should be greater than the average cost of repositioning the vehicle back to i in the next period. In particular, the condition is satisfied if the system is stationary, i.e.,

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$p_{ijt} = p_{ij}, s_{ijt} = s_{ij}$, and $p_{ij} \geq s_{ji}$—the profit of satisfying a trip from i to j is greater than the cost of repositioning the vehicle from j back to i. With this condition, formulation (2) in Lemma 1 states that even if the firm is able to hold back the vehicles and ration demands, the firm still finds it optimal to satisfy the demands in the current period to its full capability. In the subsequent discussion, we assume the above condition holds through out the rest of the paper.

Even though $V_t\left(\mathbf{x_t}, \mathbf{d_{[t-1]}}\right)$ is convex in $\mathbf{x_t}$ under some reasonable condition on the cost parameters, the state variables in (1) is still of $N + (t-1)$ dimensions and suffers from the "curse of dimensionality". Indeed, it is noted in the transshipment literature that "[o]ptimal policies for lateral transshipment generally can only be found for a small number of locations due to the large dimensions involved in multiple locations"(p.134 Paterson et al. 2011) and many works focus solely on the case of two locations (see, for example, Tagaras 1989, Chen et al. 2015, Abouee-Mehrizi et al. 2015). In the following, we first demonstrate in Section 3.1 that the optimal repositioning policy in a 2-region system with temporally independent demands has a simple structure and then discuss practical solution approaches for the general N-region system with possibly dependent demands in Section 4.

3.1. Optimal Repositioning Policy in a 2-Region System

In this section, we consider a system of two regions 1 and 2. For period t, it is sufficient to use r_t to denote the repositioned vehicles from region 1 to 2 if $r_t > 0$ and from region 2 to 1 if $r_t < 0$, because repositioning in both directions simultaneously is never optimal. Consequently, there are $r_t^+ = r_t \vee 0$ vehicles repositioned from region 1 to 2 and $r_t^- = -(r_t \wedge 0)$ vehicles repositioned from region 2 to 1. Let s_{12t} and s_{21t} denote the repositioning cost per trip from 1 to 2 and 2 to 1 respectively and the average lost sales penalties be $\bar{p}_{1t} = p_{11t}\alpha_{11t} + p_{12t}\alpha_{12t}$ and $\bar{p}_{2t} = p_{21t}\alpha_{21t} + p_{22t}\alpha_{22t}$. We assume throughout this section that the demands are independent over periods, and the condition in Lemma 1 is satisfied.

Note that the total fleet size $x_{1t} + x_{2t}$ at any period t is a constant, which we denote by C. Hence, we can reduce the dimension of the states by only using the number of available vehicles at region 1 as the state variable, i.e., we let $x_t = x_{1t}$ and keep in mind that $x_{2t} = C - x_t$. Let $y_t = x_t - r_t$ be the the number of available vehicles at region 1 *after repositioning.* We characterize the optimal repositioning policy as follows.

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Proposition 1. *Suppose $\bar{p}_{it} \geq s_{ji(t+1)}\alpha_{ijt}$ for $i, j \in \{1,2\}$ and $j \neq i$. For each period t, there exist $\underline{x}_t$ and $\overline{x}_t$ such that*

$$r_t^*(x_t) = \begin{cases} x_t - \underline{x}_t, & x_t \in [0, \underline{x}_t), \\ 0, & x_t \in [\underline{x}_t, \overline{x}_t], \\ x_t - \overline{x}_t, & x_t \in (\overline{x}_t, C], \end{cases} \quad y_t^*(x_t) = \begin{cases} \underline{x}_t, & x_t \in [0, \underline{x}_t), \\ x_t, & x_t \in [\underline{x}_t, \overline{x}_t], \\ \overline{x}_t, & x_t \in (\overline{x}_t, C]. \end{cases}$$

We call $\underline{x}_t$ and $\overline{x}_t$ the optimal reposition up-to and down-to levels respectively that are defined by the following two convex programs

$$\underline{x}_t = \arg\min_{0 \leq y \leq C} \{s_{21t} y + \mathbb{E}_{\mathbb{P}}[J_t(y, \mathbf{d_t})]\}, \quad \overline{x}_t = \arg\min_{0 \leq y \leq C} \{-s_{12t} y + \mathbb{E}_{\mathbb{P}}[J_t(y, \mathbf{d_t})]\}.$$

Proof: Please see Appendix A.2 in the Online Supplement.

The optimal repositioning policy is very intuitive and resembles the base stock policy in the traditional inventory literature. Such two-threshold policy is also well-known in the cash balance literature (see, for example, Eppen and Fama 1969). While Eppen and Fama (1969) considers a backlogging model which leads to a linear state transition, here we establish the two-threshold policy for a lost-sales model with nonlinear state transition. When the inventory level at location 1 is below the threshold $\underline{x}_t$, it is optimal to reposition vehicles from 2 to 1 and bring up the inventory level to the threshold $\underline{x}_t$; when the inventory level at location 1 is above the threshold $\bar{x}_t$, it is optimal to reposition vehicles from 1 to 2 and bring down the inventory level to the threshold $\bar{x}_t$. Within the "comfort zone" $[\underline{x}_t, \bar{x}_t]$, it is optimal to do nothing. As one might expect, as the repositioning cost at period t becomes higher, the set $[\underline{x}_t, \bar{x}_t]$ becomes larger, i.e., the firm is more likely to find it optimal to do nothing. This intuition is confirmed in the following result.

Corollary 1. *Suppose $\bar{p}_{it} \geq s_{ji(t+1)}\alpha_{ijt}$ for $i, j \in \{1,2\}$ and $j \neq i$. For each period t, $\underline{x}_t$ is decreasing in s_{21t} and $\bar{x}_t$ is increasing in s_{12t}.*

Proof: Please see Appendix A.3 in the Online Supplement.

When $T = 1$, we can further characterize the optimal repositioning policy in closed-form as the following corollary shows. For clarity, we drop t in the subscript.

Corollary 2. *When $T = 1$, let $\bar{F}_i(\cdot)$ be the survival function of d_i for $i = 1, 2$ and let $\underline{x}_0$ and $\overline{x}_0$ be the solution to the following equations respectively*

$$s_{21} + \bar{p}_1 \bar{F}_1(y) + \bar{p}_2 \bar{F}_2(C - y) = 0, \quad -s_{12} + \bar{p}_1 \bar{F}_1(y) + \bar{p}_2 \bar{F}_2(C - y) = 0.$$

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Then the optimal reposition up-to and down-to levels are respectively $\underline{x} = \underline{x}_0^+ \wedge C$ *and* $\overline{x} = \overline{x}_0^+ \wedge C$.

Proof: Please see Appendix A.4 in the Online Supplement.

One can see from Corollary 2 that the optimal repositioning policy and hence the optimal expected cost in this case only depend on the marginal distributions of d_1 and d_2, but not the joint distribution. That is, the optimal repositioning policy is independent of demand correlations across regions, when the planning horizon is a single period. When $T > 1$, however, there are numerical examples showing that this observation is no longer true.

To conclude this section, we remark that the assumption of temporal independence among demands and the assumption of a 2-region system are critical for our characterization of the simple structure of the optimal repositioning policy in Proposition 1. Indeed, Benjaafar et al. (2017) demonstrate that for multi-region system, the optimal repositioning policy can no longer be characterized by simple thresholds. However, the analysis of such simple system can help us understand the performance of approximate algorithms in general systems, which we discuss next.

4. Distributionally Robust Solution Approaches

As we have pointed out earlier, the stochastic dynamic program approach generally suffers from the "curse of dimensionality". While there are numerous approximation schemes proposed in the literature that can get around the computational difficulty, we remark that in practice the more prominent issue is to estimate the joint distribution of $\mathbf{d}_{[T]}$, the demands at all regions and all periods. Such information is usually difficult, if not impossible, to acquire in practice. We take the example of car2go mentioned in the introduction. The service region of car2go in San Diego, California consisted of 16 zip codes. Even if we crudely cluster the zip codes into $N = 5$ regions and consider a daily operations of $T = 3$ repositioning periods, $\mathbf{d}_{[T]}$ is a random vector of dimension 15. In contrast, with one-year operating data of car2go, one has only around 250 weekday demand samples, and if monthly seasonality is further taken into account, there are only around 22 weekday samples for each month. Though car2go may face hundreds of thousands of transactions each year, it has limited information when it comes to demand estimation. In this section, we address the issue of limited information by using the framework of distributionally robust optimization (DRO), where instead of assuming perfect knowledge of the distribution, one assumes the

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distribution lies in a certain ambiguity set. The resulting multi-period robust optimization problem is still computationally demanding. We propose approximate solutions based on the idea of linear decision rules to address the issue of computational tractability.

4.1. Ambiguity Set

Let $\mathcal{R}^{N,M}$ be the space of all measurable functions from $\mathbb{R}^N$ to $\mathbb{R}^M$ that are bounded on compact sets. Similar to Section 3, we define $\mathbf{d}_{[\mathbf{t}]} = (\mathbf{d_1}, \ldots, \mathbf{d_t})$, where $\mathbf{d_t} = (d_{it})$ and $\mathbf{d}_{[0]} = \emptyset$. Hence, $\mathbf{d}_{[\mathbf{t}]}$ records all demand realizations from period 1 to t. Let $\mathcal{P}_0\left(\mathbb{R}^{NT}\right)$ be the set of all distributions on a random vector of length $N \times T$ and $\mathcal{W}$ be the support of $\mathbf{d}_{[T]}$. Instead of assuming a perfect knowledge of the joint distribution of $\mathbf{d}_{[T]}$, denoted as $\mathbb{P} \in \mathcal{P}_0\left(\mathbb{R}^{NT}\right)$, we assume here that $\mathbb{P}$ lies in a distributional ambiguity set $\mathbb{F} \subset \mathcal{P}_0\left(\mathbb{R}^{NT}\right)$ that is specified by the partial distributional information estimated from data.

The choice of the ambiguity set $\mathbb{F}$ greatly affects the tractability of robust formulation. As noted in Bertsimas et al. (2018), using a second-order conic (SOC) representable ambiguity set, one can formulate the DRO problem as a second-order cone program (SOCP) that is efficiently solvable by commercial solvers including CPLEX and Gurobi. Therefore, we adopt the SOC representable ambiguity set defined below:

$$\mathbb{F} = \left\{ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}^{NT}) \;\middle|\; \begin{array}{ll} \mathbb{E}_{\mathbb{P}}(\mathbf{d}) = \boldsymbol{\mu} & \\ \mathbb{E}_{\mathbb{P}}\left((d_{it} - \mu_{it})^2\right) \le \sigma_{it}^2, & \forall i \in [N], t \in [T] \\ \mathbb{E}_{\mathbb{P}}\left(\left(\sum_{l=k}^{t} \mathbf{1}'(\mathbf{d}_l - \boldsymbol{\mu}_l)\right)^2\right) \le \gamma_{kt}^2, & \forall k, t \in [T], k \le t \\ \mathbb{P}\left(\mathbf{d} \in (\underline{\mathbf{d}}, \bar{\mathbf{d}})\right) = 1 & \end{array} \right\}.$$

The proposed ambiguity set is SOC representable as the support set $\mathcal{W} = \left[\underline{\mathbf{d}}, \bar{\mathbf{d}}\right]$ is a SOC representable set and the functions inside the expectations are SOC representable functions, e.g., $g(d_{it}) = (d_{it} - \mu_{it})^2$. Such ambiguity set requires simple descriptive statistics from data and allows us to model a rich variety of structural information about the random demands. First, it is natural to incorporate the bounded support, i.e., $\mathbb{P}\left(\mathbf{d} \in \left[\underline{\mathbf{d}}, \bar{\mathbf{d}}\right]\right) = 1$, the first moment information, i.e., $\mathbb{E}_{\mathbb{P}}(\mathbf{d}) = \boldsymbol{\mu}$, and the second moment information, i.e., $\mathbb{E}_{\mathbb{P}}\left((d_{it} - \mu_{it})^2\right) \le \sigma_{it}^2$. Furthermore, the partial cross moment information, i.e., $\mathbb{E}_{\mathbb{P}}\left(\left(\sum_{l=k}^{t} \mathbf{1}'(\mathbf{d}_l - \boldsymbol{\mu}_l)\right)^2\right) \le \gamma_{kt}^2$, captures both the demand correlations across regions and time periods that are generally observed in travel patterns.

Incorporating partial cross moment information in the ambiguity set is motivated by our data analysis of car2go's operations. From their vehicle status data between March and

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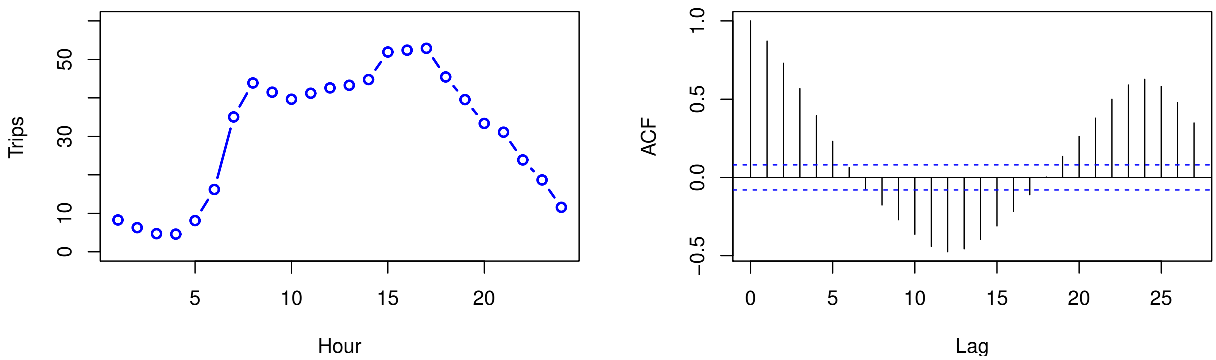
April in 2014, we summarize the travel patterns of the system-wide hourly trips in Figure 1. As we can notice from Figure 1 (a), the travel demand increases in the morning and decreases in the evening. The temporal correlations of hourly trips are shown in Figure 1 (b). The positive autocorrelations at lag 1 and 24 indicates that the demands of the next hour and the same hour tomorrow are positively correlated with the current demand. Therefore, our observation suggests the importance of incorporating demand temporal dependence in the optimization model.



(a) Average system-wide hourly trips (b) The autocorrelation function (ACF) of hourly trips

Figure 1 Travel patterns of hourly trips in car2go San Diego

Following Bertsimas et al. (2018), we introduce auxiliary random variables $\mathbf{u}, \mathbf{v}$ and consider the following so-called "lifted ambiguity set" $\mathbb{G}$:

$$\mathbb{G} = \left\{ \mathbb{Q} \in \mathcal{P}_0\left(\mathbb{R}^{NT} \times \mathbb{R}^{NT} \times \mathbb{R}^{\frac{T(T+1)}{2}}\right) \;\middle|\; \begin{array}{ll} \mathbb{E}_{\mathbb{Q}}(\mathbf{d}) = \boldsymbol{\mu} & \\ \mathbb{E}_{\mathbb{Q}}(u_{it}) \le \sigma_{it}^2, & \forall i \in [N], t \in [T] \\ \mathbb{E}_{\mathbb{Q}}(v_{kt}) \le \gamma_{kt}^2, & \forall k, t \in [T], k \le t \\ \mathbb{Q}\left((\mathbf{d}, \mathbf{u}, \mathbf{v}) \in \bar{\mathcal{W}}\right) = 1 & \end{array} \right\}.$$

Here, $\bar{\mathcal{W}}$ is the "lifted support set" defined as

$$\bar{\mathcal{W}} = \left\{ (\mathbf{d}, \mathbf{u}, \mathbf{v}) \in \mathbb{R}^{NT} \times \mathbb{R}^{NT} \times \mathbb{R}^{\frac{T(T+1)}{2}} \;\middle|\; \begin{array}{ll} \underline{\mathbf{d}} \le \mathbf{d} \le \bar{\mathbf{d}} & \\ (d_{it} - \mu_{it})^2 \le u_{it} \le \bar{u}_{it}, & \forall i \in [N], t \in [T] \\ \left(\sum_{l=k}^{t} \mathbf{1}'(\mathbf{d}_l - \boldsymbol{\mu}_l)\right)^2 \le v_{kt} \le \bar{v}_{kt}, & \forall k, t \in [T], k \le t \end{array} \right\},$$

where $\bar{u}_{it} = \max\left\{(\underline{d}_{it} - \mu_{it})^2, (\bar{d}_{it} - \mu_{it})^2\right\}$ and $\bar{v}_{kt} = \max\left\{\left(\sum_{l=k}^{t} \mathbf{1}'\left(\underline{\mathbf{d}}_l - \boldsymbol{\mu}_l\right)\right)^2, \left(\sum_{l=k}^{t} \mathbf{1}'\left(\bar{\mathbf{d}}_l - \boldsymbol{\mu}_l\right)\right)^2\right\}$.

Proposition 2. *The set of marginal distributions of* $\mathbf{d}$ *under* $\mathbb{Q}$ *for all* $\mathbb{Q} \in \mathbb{G}$*, i.e.,* $\Pi_{\mathbf{d}}\mathbb{G}$*, is equivalent to the ambiguity set* $\mathbb{F}$*. That is,* $\mathbb{F} = \Pi_{\mathbf{d}}\mathbb{G}$.

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Note that $\mathbb{G}$ is the set of distributions on the random vector $(\mathbf{d}, \mathbf{u}, \mathbf{v})$ while our original ambiguity set $\mathbb{F}$ is the set of distributions only on $\mathbf{d}$. Proposition 2 extends the lifting theorems in Bertsimas et al. (2018) by incorporating the upper bounds for auxiliary variables $\mathbf{u}$ and $\mathbf{v}$. As we show in Section 5.4, including upper bounds for $\mathbf{u}$ and $\mathbf{v}$ in the lifted ambiguity set significantly improves the performance of our proposed robust solutions.

4.2. "Myopic" Robust Solution

We first examine a single period problem with $T = 1$. The solution to the single period problem can also serve as the "myopic" solution for the multi-period problem, in which the firm only considers the repositioning cost and lost sales penalty in the current period and ignores all future costs. This simple robust model serves both as an illustration of various concepts and techniques used in DRO as well as a benchmark for our algorithm on the general multi-period model. For clarity, we temporarily drop the index t in the notations.

Given an ambiguity set of probability distributions $\mathbb{F}$, the firm seeks to minimize the worst-case expected cost over $\mathbb{F}$. That is,

$$\begin{aligned} \min_{r_{ij} \geq 0} & \sum_{i,j \in [N]} s_{ij} r_{ij} + \sup_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left[\sum_{i \in [N]} \bar{p}_i \left(d_i - w_i(\mathbf{d}) \right) \right] \\ \text{s.t.} & \sum_{j \in [N]} r_{ij} \leq x_i, \forall i \in [N] \\ & \quad w_i(\mathbf{d}) \leq d_i \wedge \left(x_i + \sum_{j \in [N]} r_{ji} - \sum_{j \in [N]} r_{ij} \right), \forall \mathbf{d} \in \mathcal{W}, i \in [N]. \end{aligned} \tag{3}$$

One can alternatively view (3) as a two stage problem, where $\mathbf{r} = (r_{ij})$ is the *"here-and-now"* repositioning decision in the first stage before the realization of demands and $w_i(\mathbf{d})$ is the adaptive decision after demand realization in the second stage. Similar to Lemma 1, we can interpret $w_i(\mathbf{d})$ as the decision of how much demand to fulfill given the realization $\mathbf{d}$ and since no future cost is considered, it is optimal to fulfill as much demand as possible. With the lifted ambiguity set, we transform the two-stage DRO problem (3) in Lemma 2.

LEMMA 2. *The two-stage distributionally robust optimization problem* (3) *is equivalent to the following optimization problem:*

$$\min_{\substack{\lambda, \boldsymbol{\eta} \\ \boldsymbol{\theta}, \mathbf{r} \geq \mathbf{0}, \delta \geq 0}} \quad \sum_{i,j \in [N]} s_{ij} r_{ij} + \lambda + \boldsymbol{\eta}' \boldsymbol{\mu} + \sum_{i \in [N]} \sigma_i^2 \theta_i + \gamma^2 \delta \tag{4}$$

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$$\text{s.t.}\quad \lambda + \boldsymbol{\eta}'\mathbf{d} + \sum_{i\in[N]} \theta_i u_i + \delta v \geq \sum_{i\in[N]} \bar{p}_i \left(d_i - x_i - \sum_{j\in[N]} r_{ji} + \sum_{j\in[N]} r_{ij} \right)^+, \forall(\mathbf{d},\mathbf{u},v) \in \bar{\mathcal{W}}$$

$$\sum_{j\in[N]} r_{ij} \leq x_i, \forall i \in [N]$$

Proof: Please see Appendix A.6 in the Online Supplement.

The optimization problem in (4) is not yet directly solvable, due to the sums of piecewise linear functions and infinite number of constraints. We further reformulate it into a solvable convex optimization problem in Proposition 3.

PROPOSITION 3. *Let $\mathcal{P}(N)$ be the power set of N. The exact optimal solution to the two-stage model in* (3) *can be obtained by solving the following SOCP:*

$$\min_{\substack{\mathbf{r},\mathbf{y},\lambda,\boldsymbol{\eta},\boldsymbol{\theta},\delta \\ \bar{\boldsymbol{\beta}}(S),\underline{\boldsymbol{\beta}}(S),\boldsymbol{\beta}(S),\beta_0(S),\boldsymbol{\rho}_i(S),\boldsymbol{\rho}_0(S)}} \sum_{i,j\in[N]} s_{ij} r_{ij} + \lambda + \boldsymbol{\eta}'\boldsymbol{\mu} + \sum_{i\in[N]} \sigma_i^2 \theta_i + \gamma^2 \delta \tag{5}$$

$$\text{s.t.}\quad y_i = x_i + \sum_{j\in[N]} r_{ji} - \sum_{j\in[N]} r_{ij}, \forall i \in [N]$$

$$\lambda + \sum_{i\in S} \bar{p}_i y_i \geq \bar{\boldsymbol{\beta}}(S)'\bar{\mathbf{d}} - \underline{\boldsymbol{\beta}}(S)'\underline{\mathbf{d}} + \frac{1}{2}\mathbf{1}'\boldsymbol{\beta}(S) + \frac{1}{2}\beta_0(S) - \sum_{i\in[N]} \boldsymbol{\rho}_i(S)'\mathbf{b}_i - \boldsymbol{\rho}_0(S)\mathbf{b}_0, \forall S \in \mathcal{P}(N)$$

$$\begin{pmatrix} \boldsymbol{\eta} - \bar{\mathbf{p}}(S) \\ \boldsymbol{\theta} \\ \delta \end{pmatrix} = \begin{pmatrix} \underline{\boldsymbol{\beta}}(S) - \bar{\boldsymbol{\beta}}(S) \\ \mathbf{0} \\ 0 \end{pmatrix} + \sum_{i\in[N]} (\mathbf{A}_i'\boldsymbol{\rho}_i(S) + \beta_i(S)\mathbf{c}_i) + \mathbf{A}_0'\boldsymbol{\rho}_0(S) + \beta_0\mathbf{c}_0, \forall S \in \mathcal{P}(N)$$

$$\|\boldsymbol{\rho}_i(S)\|_2 \leq \beta_i(S), \forall i \in [N], \forall S \in \mathcal{P}(N)$$

$$\|\boldsymbol{\rho}_0(S)\|_2 \leq \beta_0(S), \forall S \in \mathcal{P}(N)$$

$$\sum_{j\in[N]} r_{ij} \leq x_i, \forall i \in [N]$$

$$\boldsymbol{\theta} \geq \mathbf{0}, \delta \geq 0, \mathbf{r} \geq \mathbf{0}, \bar{\boldsymbol{\beta}}(S), \underline{\boldsymbol{\beta}}(S) \geq \mathbf{0}, \forall S \in \mathcal{P}(N)$$

where $\mathbf{A}_0 = \begin{bmatrix} \mathbf{1}' & \mathbf{0}' & 0 \\ \mathbf{0}' & \mathbf{0}' & \frac{1}{2} \end{bmatrix}$, $\mathbf{b}_0 = \begin{pmatrix} \mathbf{1}'\boldsymbol{\mu} \\ \frac{1}{2} \end{pmatrix}$, $\mathbf{c}_0 = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \frac{1}{2} \end{pmatrix}$, $\mathbf{A}_i = \begin{bmatrix} \mathbf{e}_i' & \mathbf{0}' & 0 \\ \mathbf{0} & \frac{1}{2}\mathbf{e}_i' & 0 \end{bmatrix}$, $\mathbf{b}_i = \begin{pmatrix} \mu_i \\ \frac{1}{2} \end{pmatrix}$, $\mathbf{c}_i = \begin{pmatrix} \mathbf{0} \\ \frac{1}{2}\mathbf{e}_i \\ 0 \end{pmatrix}$, $\forall i \in [N]$, $\bar{\mathbf{p}}(S) = (\bar{p}_i(S))_{i\in[N]}$ *with* $\bar{p}_i(S) = \bar{p}_i$ *if* $i \in S$ *and* 0 *otherwise.*

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Proof: Please see Appendix A.7 in the Online Supplement.

Our myopic robust model is now readily solvable by commercial solvers, e.g., CPLEX and Gurobi, using the SOCP formulation in Proposition 3. However, both the number of decision variables as well as constraints grows exponentially in N, due to $\mathcal{P}(N)$. Therefore, using the above formulation, one can only obtain the exact robust solutions for small size instances. As noted in Ardestani-Jaafari and Delage (2016), problems involving the sum of piecewise linear functions are generally NP-hard, and one commonly used technique is to approximate the piecewise linear functions via linear decision rule. Indeed, we apply the idea of linear decision rule in designing our heuristic for the multi-period problem and we will show in Section 4.4 below that our approximation algorithm gives the exact optimal solution to the single period problem considered here under a mild technical condition.

4.3. Multi-Period Robust Solution

As we have already seen in Section 4.2, even in a single period problem, the adaptive decisions are piecewise-linear functions of the realized demand, and the exact reformulation of the robust model in (5) has exponential number of variables and constraints. In the multi-period problem, future repositioning r_{ijt}, system state x_{it} and demand fulfillment w_{ijt} all become adaptive decisions, which can be general functions of all past demand realizations, and hence the problem becomes even harder. We emphasize that, the adaptive decisions we consider are *non-anticipative*, i.e., the adaptive decisions r_{ijt}, x_{it}, w_{ijt} can be represented as functions $r_{ijt}(\mathbf{d}_{[\mathbf{t-1}]})$, $x_{it}(\mathbf{d}_{[\mathbf{t-1}]})$ and $w_{it}(\mathbf{d}_{[\mathbf{t}]})$. Notice here that w_{it} depends on $\mathbf{d}_{[\mathbf{t}]}$ while r_{ijt} and x_{it} depend on $\mathbf{d}_{[\mathbf{t-1}]}$ since the former decision is made after the demand at period t realizes while the latter are made before that.

To approximate the adaptive decisions, a common technique known as the linear decision rule (LDR) is to restrict such decisions to affine functions of the random variables. In our problem, we can apply the LDR by restricting $r_{ijt}(\mathbf{d}_{[\mathbf{t-1}]})$, $x_{it}(\mathbf{d}_{[\mathbf{t-1}]})$ and $w_{it}(\mathbf{d}_{[\mathbf{t}]})$ to be linear functions, that is, $\mathbf{x}_t(\cdot) \in \mathcal{L}^N(\mathbf{d}_{[\mathbf{t-1}]})$, $\mathbf{r}_t(\cdot) \in \mathcal{L}^{N^2}(\mathbf{d}_{[\mathbf{t-1}]})$, and $\mathbf{w}_t(\cdot) \in \mathcal{L}^N(\mathbf{d}_{[\mathbf{t}]}), \forall t \in [T]$, where for any positive integer M, we define

$$\mathcal{L}^M(\mathbf{d}_{[\mathbf{t}]}) = \left\{ \mathbf{y} \in \mathcal{R}^{Nt,M} \;\middle|\; \exists \mathbf{y}^0, \mathbf{y}^1_{il} \in \mathbb{R}^M, i \in [N], l \in [t]: \;\; \mathbf{y}(\mathbf{d}_{[\mathbf{t}]}) = \mathbf{y}^0 + \textstyle\sum_{i\in[N], l\in[t]} \mathbf{y}^1_{il} d_{il} \right\}$$

to be the set of all measurable mappings from $\mathbb{R}^{Nt}$ to $\mathbb{R}^M$ that are affinely dependent on the realized demands in all regions from period 1 to period t. In the following, we use the same

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letter to express both an affine mapping and its linear coefficients but we use superscripts on the letter to emphasize that the letter represents a particular linear coefficient. For example, in the LDR for period t, $\mathbf{x}_t^0$ denotes a vector of intercepts of the affine mapping $\mathbf{x}_t(\cdot)$, while $\mathbf{r}_{ilt}^1$ denotes a vector of slopes corresponding to d_{il} of the affine mapping $\mathbf{r}_t(\cdot)$.

An important consideration in defining the lifted ambiguity set $\mathbb{G}$ in Section 4.1 is its ability to improve the LDR approximation. Instead of restricting, say, $w_{it}(\cdot)$ to be a linear function of only $\mathbf{d}_{[\mathbf{t}]}$, one can incorporate the information from auxiliary random variables $(\mathbf{u}, \mathbf{v})$ by letting $w_{it}(\cdot)$ to be a linear function of $(\mathbf{d}_{[\mathbf{t}]}, \mathbf{u}_{[t]}, \mathbf{v}_{[t]})$, where

$$\mathbf{u}'_{[t]} = (\mathbf{u}'_1, \ldots, \mathbf{u}'_t) \text{ with } \mathbf{u}_t = (u_{it})_{i\in[N]}, \quad \mathbf{v}'_{[t]} = (v_{kl})_{1\le k\le l\le t}, \ \ \forall t \in [T].$$

More formally, we restrict $\mathbf{x}_t(\cdot) \in \bar{\mathcal{L}}^N(\mathbf{d}_{[\mathbf{t-1}]}, \mathbf{u}_{[t-1]}, \mathbf{v}_{[t-1]})$, $\mathbf{r}_t(\cdot) \in \bar{\mathcal{L}}^{N^2}(\mathbf{d}_{[\mathbf{t-1}]}, \mathbf{u}_{[t-1]}, \mathbf{v}_{[t-1]})$, and $\mathbf{w}_t(\cdot) \in \bar{\mathcal{L}}^N(\mathbf{d}_{[\mathbf{t}]}, \mathbf{u}_{[t]}, \mathbf{v}_{[t]})$, where for any positive integer M, we define

$$\bar{\mathcal{L}}^M(\mathbf{d}_{[\mathbf{t}]}, \mathbf{u}_{[t]}, \mathbf{v}_{[t]}) = \left\{ \mathbf{y} \in \mathcal{R}^{2Nt+\frac{t(t+1)}{2}, M} \;\middle|\; \begin{array}{c} \exists \mathbf{y}^0, \mathbf{y}_{il}^1, \mathbf{y}_{il}^2, \mathbf{y}_{kl}^3 \in \mathbb{R}^M, \\ i\in[N], k\in[l], l\in[t] \end{array} : \begin{array}{r} \mathbf{y}(\mathbf{d}_{[\mathbf{t}]}, \mathbf{u}_{[t]}, \mathbf{v}_{[t]}) = \mathbf{y}^0 + \\ \sum_{i\in[N], l\in[t]} \mathbf{y}_{il}^1 d_{il} + \\ \sum_{i\in[N], l\in[t]} \mathbf{y}_{il}^2 u_{il} + \\ \sum_{l\le t} \sum_{k\le l} \mathbf{y}_{kl}^3 v_{kl} \end{array} \right\}$$

to be the set of all measurable functions from $\mathbb{R}^{2Nt+\frac{t(t+1)}{2}}$ to $\mathbb{R}^M$ that are affinely dependent on all the revealed information from period 1 to period t, i.e., $\mathbf{d}_{[\mathbf{t}]} \in \mathbb{R}^{Nt}, \mathbf{u}_{[t]} \in \mathbb{R}^{Nt}$ and $\mathbf{v}_{[t]} \in \mathbb{R}^{\frac{t(t+1)}{2}}$. The decision rule described above that utilizes the auxiliary information in the lifted ambiguity set is also referred to as the *enhanced linear decision rule* (ELDR) in the literature and is empirically observed to have a much better performance than its LDR counterpart (see Bertsimas et al. 2018).

Having defined the mapping functions for adaptive decisions, we now provide the multi-period DRO model that minimizes the worst-case expected total cost over the entire horizon, under the lifted ambiguity set $\mathbb{G}$ with ELDR, as follows

$$\begin{aligned} \min_{r_{ij1}\ge 0} \quad & \sum_{i\in[N]} \sum_{j\in[N]} s_{ij1} r_{ij1} + F(\mathbf{y}_1) \\ \text{s.t.} \quad & y_{i1} = x_{i1} + \sum_{j\in[N]} r_{ji1} - \sum_{j\in[N]} r_{ij1}, \forall i \in [N] \\ & \sum_{j\in[N]} r_{ij1} \le x_{i1}, \forall i \in [N] \end{aligned} \tag{6}$$

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where

$$F(\mathbf{y}_1) = \min \sup_{\mathbb{Q}\in\mathbb{G}} \mathbb{E}_{\mathbb{Q}} \left[\sum_{t\in[T]} \sum_{i\in[N]} \bar{p}_{it} \left(d_{it} - w_{it}(\cdot) \right) + \sum_{t\in[T-1]} \sum_{i,j\in[N]} s_{ij(t+1)} r_{ij(t+1)}(\cdot) \right] \tag{7}$$

$$\text{s.t.} \sum_{j\in[N]} r_{ij(t+1)}(\cdot) \le x_{i(t+1)}(\cdot), \forall(\mathbf{d},\mathbf{u},\mathbf{v}) \in \bar{\mathcal{W}}, i\in[N], t\in[T-1]$$

$$0 \le r_{ij(t+1)}(\cdot), \forall(\mathbf{d},\mathbf{u},\mathbf{v}) \in \bar{\mathcal{W}}, i,j\in[N], t\in[T-1]$$

$$x_{i(t+1)}(\cdot) = x_{it}(\cdot) + \sum_{j\in[N]} \left(\alpha_{jit} w_{jt}(\cdot) + r_{jit}(\cdot) - \alpha_{ijt} w_{it}(\cdot) - r_{ijt}(\cdot) \right), \forall(\mathbf{d},\mathbf{u},\mathbf{v}) \in \bar{\mathcal{W}}, i\in[N], t\in[T-1]$$

$$y_{i(t+1)} = x_{i(t+1)}(\cdot) + \sum_{j\in[N]} r_{ji(t+1)}(\cdot) - \sum_{j\in[N]} r_{ij(t+1)}(\cdot), \forall(\mathbf{d},\mathbf{u},\mathbf{v}) \in \bar{\mathcal{W}}, i\in[N], t\in[T-1]$$

$$w_{it}(\cdot) \le d_{it} \wedge y_{it}, \forall(\mathbf{d},\mathbf{u},\mathbf{v}) \in \bar{\mathcal{W}}, i\in[N], t\in[T]$$

with $\mathbf{x}_{t+1}(\cdot) \in \bar{\mathcal{L}}^N(\mathbf{d}_{[\mathbf{t}]}, \mathbf{u}_{[t]}, \mathbf{v}_{[t]})$, $\mathbf{r}_{t+1}(\cdot) \in \bar{\mathcal{L}}^{N^2}(\mathbf{d}_{[\mathbf{t}]}, \mathbf{u}_{[t]}, \mathbf{v}_{[t]})$, and $\mathbf{w}_t(\cdot) \in \bar{\mathcal{L}}^N(\mathbf{d}_{[\mathbf{t}]}, \mathbf{u}_{[t]}, \mathbf{v}_{[t]})$.

Compared to the stochastic dynamic program (2) introduced in Section 3, the multi-period DRO model in formulation (6) seeks to minimize a worst-case objective by considering all possible distribution $\mathbb{Q}$ in the ambiguity set $\mathbb{G}$. Note that if one does not restrict $w_{it}(\cdot)$ to be affine functions of $(\mathbf{d}_{[\mathbf{t}]}, \mathbf{u}_{[t]}, \mathbf{v}_{[t]})$, then Lemma 1 also guarantees here that the optimal demand fulfillment decisions are $w^*_{it}(\cdot) = d_{it} \wedge \Big(x_{it}(\cdot) + \sum_{j\in[N]} r_{jit}(\cdot) - \sum_{j\in[N]} r_{ijt}(\cdot) \Big)$, which are nonlinear functions and would result in an intractable formulation. As we will see shortly, by restricting $\mathbf{w}_t(\cdot)$ (and also $\mathbf{x}_t(\cdot), \mathbf{r}_t(\cdot)$) to be affine functions, our ELDR approximation in (6) can be solved very efficiently. Similar to Lemma 2, we can reformulate $F(\mathbf{y}_1)$ as a minimization problem below.

LEMMA 3. *The multi-period DRO problem $F(\mathbf{y}_1)$ is equivalent to:*

$$\min_{\substack{\boldsymbol{\theta},\boldsymbol{\delta}\ge\mathbf{0},\lambda,\boldsymbol{\eta} \\ \mathbf{x}^0_{t+1}, \mathbf{x}^1_{il(t+1)}, \mathbf{x}^2_{il(t+1)}, \mathbf{x}^3_{kl(t+1)} \\ \mathbf{r}^0_{t+1}, \mathbf{r}^1_{il(t+1)}, \mathbf{r}^2_{il(t+1)}, \mathbf{r}^3_{kl(t+1)} \\ \mathbf{w}^0_t, \mathbf{w}^1_{ilt}, \mathbf{w}^2_{ilt}, \mathbf{w}^3_{klt}}} \quad \lambda + \boldsymbol{\eta}'\boldsymbol{\mu} + \sum_{\substack{i\in[N]\\t\in[T]}} \sigma_i^2 \theta_i + \sum_{\substack{k,t\in[T]\\k\le t}} \gamma_{kt}^2 \delta_{kt} \tag{8}$$

s.t. *Constraints in* (7)

$$\lambda + \boldsymbol{\eta}'\boldsymbol{d} + \sum_{\substack{i\in[N]\\t\in[T]}} \theta_{it} u_{it} + \sum_{\substack{k,t\in[T]\\k\le t}} \delta_{kt} v_{kt} \ge \sum_{\substack{i\in[N]\\t\in[T]}} \bar{p}_{it} \left(d_{it} - w_{it}(\cdot) \right) + \sum_{\substack{i,j\in[N]\\t\in[T-1]}} s_{ij(t+1)} r_{ij(t+1)}(\cdot), \forall(\mathbf{d},\mathbf{u},\mathbf{v}) \in \bar{\mathcal{W}}$$

where $\mathbf{x}_{t+1}(\cdot) \in \bar{\mathcal{L}}^N(\mathbf{d}_{[\mathbf{t}]}, \mathbf{u}_{[t]}, \mathbf{v}_{[t]})$, $\mathbf{r}_{t+1}(\cdot) \in \bar{\mathcal{L}}^{N^2}(\mathbf{d}_{[\mathbf{t}]}, \mathbf{u}_{[t]}, \mathbf{v}_{[t]})$, *and* $\mathbf{w}_t(\cdot) \in \bar{\mathcal{L}}^N(\mathbf{d}_{[\mathbf{t}]}, \mathbf{u}_{[t]}, \mathbf{v}_{[t]})$.

Proof: We omit the proof here as it is similar to the proof of Lemma 2.

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The parameters that specify the ELDR are also decision variables in the first stage in order to minimize the worst-case expected total cost. Moreover, constraints in problem (8) are linear, since the adaptive decisions $(\mathbf{x}, \mathbf{r}, \mathbf{w})$ are linear in $(\mathbf{d}, \mathbf{u}, \mathbf{v}) \in \bar{\mathcal{W}}$ under the ELDR. Thus, problem (8) is a linear optimization problem with infinite constraints. Similar to Proposition 3, we can then transform the robust optimization problem (8) and subsequently problem (6) into an SOCP that can be efficiently solved by standard commercial solvers.

We would like to emphasize that in practical implementation (and in our numerical study as well) problem (6) is solved in a rolling-horizon fashion. At each period, only the *here-and-now* decision is implemented. For example, at the first period, we only implement $\mathbf{r}_1$ that is solved from problem (6), without any obligation to using $\mathbf{r}_2$ in the second period. When the first period's demands realize and one reaches the second period, problem (6) is solved again with a horizon $T-1$ and we implement the new *here-and-now* solution $\mathbf{r}_2$ solved from (6) in the second period. The rolling-horizon type implementation is also commonly used in other heuristics for dynamic optimization problems as well.

The ELDR approach proposed here not only facilitates a tractable reformulation, it also enjoys a couple of nice properties. We remark that in a 2-region system, the repositioning policy derived from the ELDR in Section 4.3 preserves the reposition up-to and down-to structure that is observed in the optimal dynamic programming solution in Proposition 1. The details of the discussion are provided in the Appendix A.9.

4.4. The Optimality of ELDR When $T = 1$

Recall that in Section 4.2, we introduce the "myopic" robust solution approach as a single period two-stage DRO problem (3), whose SOCP reformulation has exponential number of decision variables and constraints, and hence cannot be efficiently implemented when the problem size is large. Here, we show that our ELDR heuristic results in an optimal solution for problem (3) under a mild technical condition. Suppose Z^* is the optimal value to problem (3). Our ELDR approach applied to problem (3) results in the following problem:

$$
\begin{aligned}
Z^{\mathrm{ELDR}} = \min_{\substack{r_{ij} \geq 0 \\ \mathbf{w}(\cdot) \in \bar{\mathcal{L}}^N(\mathbf{d}, \mathbf{u}, v)}} & \sum_{i,j \in [N]} s_{ij} r_{ij} + \sup_{\mathbb{Q} \in \mathbb{G}} \mathbb{E}_{\mathbb{Q}} \left[\sum_{i \in [N]} \bar{p}_i \left(d_i - w_i(\mathbf{d}, \mathbf{u}, v) \right) \right] \\
\text{s.t.} & \sum_{j \in [N]} r_{ij} \leq x_i, \forall i \in [N] \\
& w_i(\mathbf{d}, \mathbf{u}, v) \leq d_i \wedge \left(x_i + \sum_{j \in [N]} r_{ji} - \sum_{j \in [N]} r_{ij} \right), \forall (\mathbf{d}, \mathbf{u}, v) \in \bar{\mathcal{W}}, i \in [N].
\end{aligned}
$$

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Note here that the formulation above restricts the recourse decisions $\mathbf{w}(\cdot)$ to be a linear function of $\mathbf{d}, \mathbf{u}$ and v, while in (3), the recourse decision $\mathbf{w}(\cdot)$ is only a function of $\mathbf{d}$ but can be any general function (in fact, by Lemma 1, we know that the optimal recourse decision is piecewise linear, i.e., $w_i^*(\mathbf{d}) = d_i \wedge (x_i + \sum_{j\in[N]} r_{ji} - \sum_{j\in[N]} r_{ij})$). Nevertheless, we can show in the following proposition that when demands among different regions are not too negatively correlated, the ELDR approach results in an optimal solution to (3).

Proposition 4. *If* $\gamma \geq \sqrt{\sum_{i\in[N]} \sigma_i^2}$, *then* $Z^{ELDR} = Z^*$.

Proof: Please see Appendix A.8 in the Online Supplement.

We remark that the technical condition $\gamma \geq \sqrt{\sum_{i\in[N]} \sigma_i^2}$ is only sufficient for $Z^{\text{ELDR}} = Z^*$. Numerically, we have also observed $Z^* = Z^{\text{ELDR}}$ for many instances with $\gamma \geq \sqrt{\sum_{i\in[N]} \sigma_i^2}$ violated. However, when γ is too small, there are counterexamples in which $Z^* < Z^{\text{ELDR}}$.

Proposition 4 adds new analytical insights to the limited literature that seeks to understand the effectiveness of applying linear decision rule to robust optimization problems. In particular, Iancu et al. (2013) provide conditions for the optimality of linear decision rule when the ambiguity set only contains support information while Bertsimas et al. (2018) prove the optimality of ELDR for two-stage problem when the second stage decision is of one dimension. However, neither result applies to problem (3), where the ambiguity set contains moments information and the second stage decision is of N dimension. By exploiting the special structure in our problem, Proposition 4 establishes that ELDR can also be optimal for two-stage DRO problem with arbitrary number of recourse decisions.

5. Numerical Studies

We conduct numerical experiments in various settings to examine the performance of multi-stage robust ELDR solution (or ELDR solution in short) and explore managerial insights. We first compare, in a 2-region system, the ELDR solution with the optimal solution from solving the dynamic programming using simulated data sets with known probability distributions. Furthermore, we discuss the impact of temporal demand correlation and the effectiveness of partial cross moment information in the ambiguity set.

To complement our simulation study, we consider the real-world application of car2go in San Diego, California. With its operational data, we illustrate the computational scalability and the solution quality of the multi-stage robust model. Using "no repositioning" policy as a benchmark, we quantify the value of repositioning under several considered

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approaches. We also examine the patterns of repositioning activities and travel demands, and explore how fleet size affects the repositioning performance. Finally, we consider two extensions to discuss the following questions: 1) can we improve the performance of ELDR by incorporating spatial-temporal correlations? and 2) how does ELDR perform when there is a capacity constraint on the repositioning quantity for each period?

All numerical experiments are carried out in CPLEX on a Windows Operating System with 3.2GHz CPU and 32GB RAM. In the experiments, we include the mean value problem (MVP) (e.g., Kek et al. 2009, Febbraro et al. 2012) as one benchmark. The MVP assumes deterministic demands as their means, i.e., $\mu_{it}, \forall i \in [N], t \in [T]$, and is solved as a linear program provided in Appendix B.1. Similar to our ELDR solution, in all numerical studies, the MVP solution is implemented in a rolling horizon fashion.

5.1. A 2-Region System

In this section, we test four different models: DP, MVP, "myopic" and ELDR in a 2-region system with total 212 vehicles. We consider the 2-region system here, because the DP has difficulty in dealing with larger systems due to the "curse of dimensionality". The planning horizon is set to $T = \{1, 2, 3, 4\}$. For each period t, the expected trip demands at region 1 and 2 are $\mu_{1t} = 176$ and $\mu_{2t} = 36$ respectively. The trip distribution, lost sales penalty per trip, and repositioning cost per trip are provided in Appendix B.2.

To evaluate the performance, we generate demands independently from identical distributions: truncated normal, uniform and Poisson. In the normal distribution, the mean is μ_{it} with the standard deviation $\sigma_{it} = \frac{1}{\sqrt{3}}\mu_{it}$. As for the uniform distribution, the range is set as $[0.5\mu_{it}, 1.5\mu_{it}]$. The mean demand μ_{it} is also used as the arrival rate in Poisson distribution. We conduct $20,000$ numerical experiments with simulated demand samples from the above probability distributions and evaluate the total cost with equal initial number of vehicles at two locations. The average total costs from MVP, "myopic" and ELDR models are then benchmarked with those from DP.

In Table 1, we report the relative difference between the average costs from MVP, "myopic", ELDR and that from DP in percentage respectively. As shown in Table 1, the ELDR solutions perform closely to the benchmarking DP solutions which are the exact optimal solutions. The average total costs under ELDR are within 6% gap and many of them are less than 2% from those under DP. It is also worthwhile to notice that the performance gaps of ELDR are stable under different distributions of demands. Notably, when

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Distribution	T	Performance Gap from DP		
		MVP	"myopic"	ELDR
Normal	4	7.51%	40.20%	5.21%
	3	10.15%	35.13%	5.33%
	2	8.12%	22.66%	1.76%
	1	20.10%	1.23%	1.23%
Poisson	4	8.35%	14.51%	4.67%
	3	6.74%	11.86%	2.63%
	2	6.51%	7.60%	1.34%
	1	7.23%	1.52%	1.52%
Uniform	4	12.80%	38.27%	3.62%
	3	9.12%	31.03%	1.20%
	2	13.37%	21.36%	3.28%
	1	27.35%	1.32%	1.32%

Table 1 **Benchmarking with DP under Normal, Uniform and Poisson distributions**

$T = 1$, the "myopic" and ELDR solutions coincide, as we have shown in Proposition 4. However, for longer horizons, both MVP and ELDR outperform the "myopic" model in which the firm only considers the current period and ignores all future information.

In the following, we introduce temporal demand correlation in the experiments under the setting of Normal demand distribution and planning horizon $T = 4$. The demand in region i at period t is generated by $d_{it} = \mu_{it} + \xi_i + \epsilon_{it}$, where $\xi_i \sim N(0, \hat{\sigma}_i^2)$ and $\epsilon_{it} \sim N(0, \sigma_i^2)$. The random variable ξ_i is region specific and time-independent. By scaling its variance $\hat{\sigma}_i^2$, i.e., $\hat{\sigma}_i^2 \in \{0.5 \times \sigma_i^2, \times\sigma_i^2, 1.5 \times \sigma_i^2, 2 \times \sigma_i^2, 2.5 \times \sigma_i^2, 3 \times \sigma_i^2\}$, the demand temporal correlation grows with larger scaling factors. To examine the effectiveness of incorporating partial cross moment information in $\mathbb{F}$, we also consider a reduced ambiguity set without constraint $\mathbb{E}_{\mathbb{P}}\left(\left(\sum_{l=k}^{t} \mathbf{1}'(\mathbf{d}_l - \boldsymbol{\mu}_l)\right)^2\right) \leq \gamma_{kt}^2$, and denote the corresponding ELDR approach by ELDR-2.

Figure 2 reports the out-of-sample average daily costs under DP, ELDR and ELDR-2 approaches. The daily costs under all approaches increase as the temporal correlation of demand strengthens when the scaling factor increases. Moreover, the performance of ELDR-2 is much less satisfactory compared to ELDR. In particular, the gap between ELDR and ELDR-2 becomes greater when the temporal correlation is higher. It indicates the importance of incorporating partial cross moment information in the ELDR approach. Notably, when the temporal demand correlation is significant, e.g., the scaling factor is larger than 2, ELDR outperforms DP, which ignores the demand temporal dependence.

5.2. Case Study of car2go San Diego

In this section, we quantify the value of repositioning using ELDR approach and two benchmarks, MVP and the "myopic" model, and compare their performances in a real case

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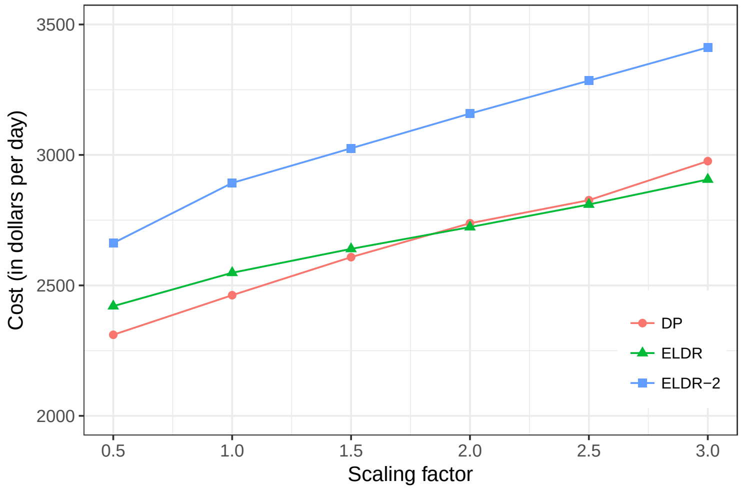

Figure 2 Performance of DP, ELDR and ELDR-2 under demand temporal correlations.

study of car2go. As we have argued in Section 4, the DP approach is not applicable in this case, since we have limited information about the distribution of the demands and it is not computationally scalable to multi-region systems with temporal dependence in demands.

We have collected the vehicle status data between March and April, 2014, at every 5-min interval, with a total of $25,873$ trips over 22 weekdays and 8 weekends identified (e.g., see Appendix B.3 for sample data). That is, we only have 22 samples of weekday demands and 8 samples of weekend demands, which is insufficient to obtain an accurate estimation of the joint distribution of $\mathbf{d}_{[T]}$. We focus on the 13 zip codes where over 90% trips were originated from and cluster them into 5 regions based on the inter-region trip intensities (e.g., see Appendix B.4 for details). We vary the system from 2-region to 5-region in the experiments, e.g., it includes regions $\{1,...,N\}$ in an $N-$region system. We divide a day into T repositioning periods with $T \in \{3,4,5\}$. Note that the demand intensity in each period also depends on T, as the period length is 6 hours when $T=4$, in contrast to 8 hours when $T=3$. In this data set, we have the origin and destination information of the completed trips in each period, which we use to approximate the demand from each origin and estimate trip distribution α_{ijt}. We can estimate the mean μ_{it}, the variance σ_{it}^2 and the cross moment γ_{kt}^2 to construct the ambiguity sets for the "myopic" and ELDR models.

To estimate cost parameters, we follow car2go's fare formula: $p_{ijt} = p \times t_{ij}$, where p is per-minute fare rate and t_{ij} is travel time from region i to region j. Similarly, we assume the repositioning cost follows the same formula with $s_{ijt} = s \times t_{ij}$, where s is the per-minute repositioning cost. The per-minute fare rate of car2go is $p = \$0.41$ (see car2go 2016). We

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N	T	p/s	MVP		"myopic"		ELDR	
			Repositioning	VoR	Repositioning	VoR	Repositioning	VoR
2	3	0.41/0.32	1.49	10.42%	1.96	31.77%	1.58	34.33%
	4		1.42	7.67%	1.81	29.48%	1.48	33.68%
	5		1.89	4.53%	1.80	8.61%	1.77	13.38%
	3	0.41/0.5	1.42	5.83%	1.64	17.13%	1.33	21.44%
	4		1.31	5.31%	1.50	15.55%	1.23	20.74%
	5		1.50	6.55%	1.55	12.80%	1.44	18.54%
3	3	0.41/0.32	2.09	7.56%	2.36	15.98%	2.21	17.87%
	4		2.03	6.83%	2.31	18.33%	2.07	21.94%
	5		2.02	6.29%	2.31	11.59%	2.11	15.35%
	3	0.41/0.5	1.96	4.97%	2.09	10.91%	1.85	17.69%
	4		1.87	4.57%	1.85	6.70%	1.71	11.93%
	5		1.92	5.71%	2.14	11.11%	1.88	12.16%
4	3	0.41/0.32	2.96	6.29%	–	–	2.92	14.38%
	4		3.75	7.50%	–	–	4.03	30.51%
	5		4.18	5.39%	–	–	4.32	16.54%
	3	0.41/0.5	2.86	3.77%	–	–	2.82	17.41%
	4		3.75	4.08%	–	–	3.86	19.77%
	5		4.18	3.09%	–	–	4.31	16.09%
5	3	0.41/0.32	4.15	4.55%	–	–	4.30	15.12%
	4		5.03	4.59%	–	–	5.23	17.37%
	5		5.72	5.67%	–	–	5.87	15.04%
	3	0.41/0.5	4.05	3.64%	–	–	4.00	14.65%
	4		4.80	3.94%	–	–	4.67	15.71%
	5		5.35	2.95%	–	–	5.10	15.10%

Table 2 **Average daily repositioning frequency and VoR under MVP, "myopic" and ELDR.**

consider two possible per-minute repositioning cost with $s \in \{\$0.32, \$0.50\}$. The travel time t_{ij} is measured using ArcGIS based on the road network in San Diego. The total fleet size C is then set to be the maximum demand within a day such that the lost sales mainly comes from the imbalance of vehicle distribution rather than insufficient capacity.

To evaluate the performance of MVP, "myopic" and ELDR solutions, we use bootstrap sampling to generate $1,000$ weekday demand samples and 400 weekend demand samples from the 22 weekday demands and 8 weekend demands in our data. For each of the sample, we compute the total cost over T periods from all solutions. In addition, we compute for each solution the total repositioning frequencies, defined as the total number of non-zero repositioning operations over all regions and periods, i.e., $\sum_{i,j,t} 1_{\{r_{ijt}>0\}}$.

We use the "no repositioning" scenario, i.e., $r_{ijt} = 0, \forall i,j \in [N], t \in T$, as the reference, and define the term "value of repositioning" (VoR) by the cost reduction in percentage, from implementing repositioning. In Table 2, we summarize the average daily repositioning frequency and VoR under the following approaches: MVP, "myopic" and ELDR. Also, the average computation time of the respective models is reported in Table 3.

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N	T	MVP	"myopic"	ELDR
2	3	< 1	< 1	< 1
2	4	< 1	< 1	< 1
2	5	< 1	< 1	< 1
3	3	< 1	5	< 1
3	4	< 1	9	< 1
3	5	< 1	13	< 1
4	3	< 1	2,380	1
4	4	< 1	3,661	1
4	5	< 1	6,799	2
5	3	< 1	–	10
5	4	< 1	–	18
5	5	< 1	–	23

Table 3 Average CPU Time (in seconds) of MVP, "myopic" and ELDR

We notice several advantages of ELDR compared with MVP and the "myopic" model: 1) The ELDR solutions bring significant VoR from 11.93% to 34.33%, higher than MVP and "myopic" with similar reposition frequency; 2) The ELDR model is computationally scalable to handle larger systems, while the "myopic" model takes more than an hour to solve for a 4-region system; and 3) The VoR of ELDR is generally higher when the repositioning cost outweighs the lost sales penalty.

Furthermore, VoR generally reduces when N is larger. With more regions, the system may become more balanced with customer trips on more origin-destination pairs. For instance, including region 4 to the 3-region system allows customers to travel from and to the newly added region, which may help the system self-balance via customer trips.

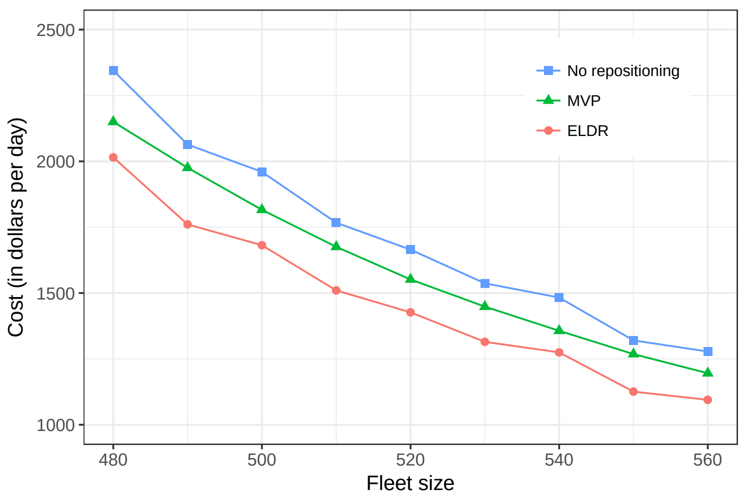

Figure 3 Impact of fleet size on the performance of no repositioning, MVP and ELDR approaches

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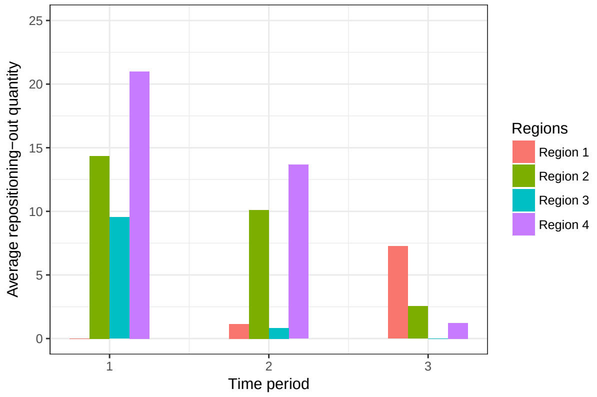
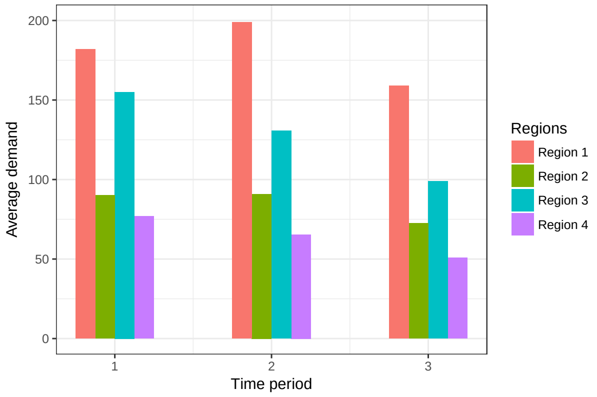
We further conduct numerical experiments on the case of $N = 4$ and $T = 3$. Since the "myopic" model is not scalable and less satisfactory than ELDR, we exclude it in the subsequent discussion. Figure 3 suggests that larger fleet size helps alleviate the operating cost by meeting more demand. Nevertheless, proactive fleet repositioning is still beneficial, even with large fleet size. The gap between "no repositioning" and ELDR, i.e., the VoR of ELDR, is consistently in the range from 14.04% to 14.74%.



(a) Repositioning pattern on weekday



(b) Demand pattern on weekday

Figure 4 Repositioning and demand patterns on weekday

In Figure 4, we also depict the connection between the repositioning and demand patterns. Figure 4 (a) reports the average repositioning-out quantities in each region i, summarized by $\sum_{j\in[N]} r_{ijt}$, using ELDR among all experiment instances. For example, it suggests that, on average, the firm repositions vehicles from regions 2, 3 and 4 into region 1 in period 1, and repositions vehicles from regions 1, 2 and 4 into region 3 in the last period. Note that in one experiment instance, it is never optimal for all regions to reposition vehicles out in the same period. However, after averaging over all experiment instances, the average repositioning-out quantities may be positive for all regions at some period.

By comparing the patterns of repositioning decisions and travel demands, it is clear that, the firm should relocate vehicles to high-demand regions in the early periods, e.g., region 1 in Figure 4 (a). Meanwhile, the repositioning activity becomes milder in later periods. It is because that repositioning in the early periods not only satisfies more demands in the current period but also improves value-to-go and system dynamics in the future periods. The repositioning decisions in the last period only depend on the system state as well as the demand distribution in a single-period problem setting.

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5.3. Extension: spatial-temporal correlation

We extend the proposed form of ambiguity set $\mathbb{F}$, as well as the "lifted ambiguity set" $\mathbb{G}$, to include the spatial-temporal correlations. We observe that, for example, the demand in region 1 at time 2 has a positive correlation of 0.513 with the demand in region 2 at time 3. One possible reason is that some customers who drive from region 1 (to region 2) at time 2 may return from region 2 (to region 1) at time 3.

We refer to the ELDR approach with the original ambiguity set $\mathbb{F}$ as the basic setting here. We then consider two extended settings for ambiguity sets. In the first setting, we extend the ambiguity set to $\mathbb{F}_1$ that includes additional pairwise correlation information, e.g., covariances ϕ^2_{ikjt}, whose correlation coefficients are greater than 0.5. In the second setting, we extend the ambiguity set to $\mathbb{F}_2$ that includes all pairwise correlation information. Their detailed formulations are provided in Appendix B.5.

p/s	Ambiguity set	VoR	Repositioning	CPU time (in seconds)
0.41/0.32	$\mathbb{F}$	14.39%	2.92	1
	$\mathbb{F}_1$	14.94%	2.92	12
	$\mathbb{F}_2$	15.30%	2.91	36
0.41/0.5	$\mathbb{F}$	17.42%	2.82	2
	$\mathbb{F}_1$	18.13%	2.81	22
	$\mathbb{F}_2$	18.33%	2.83	36

Table 4 **ELDR performance in ambiguity sets with spatial-temporal demand correlation information**

We report their performance in Table 4, for the case with $N = 4$ and $T = 3$. Under both cost structures, incorporating more spatial-temporal correlation information helps the ELDR model the demand distribution more precisely, and thus enhances its VoR with similar frequency of repositioning. Nevertheless, including all pairwise spatial-temporal correlations improves the VoR by only 1% from the basic setting $\mathbb{F}$, at the cost of significantly increasing the computation time. Therefore, when implementing ELDR approach, the firm should trade-off between the performance and the computational efficiency.

5.4. Extension: capacitated repositioning

One challenge in fleet repositioning is that it may be constrained by limited manpower resources, as such operations require manual efforts to move the vehicles. In this extension, we examine the effectiveness of ELDR in dealing with repositioning capacity R_t for each period t. We directly incorporate the constraint $\sum_{i,j\in[N]} r_{ijt} \le R_t, \forall t \in [T]$ to the ELDR problem (6). The performance of ELDR developed in this paper is benchmarked with the

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ELDR approach developed in Bertsimas et al. (2018), which is referred to as ELDR-BSZ. As discussed in Section 4.1, the key difference is that, compared to ELDR-BSZ, we extend the "lifted support set" $\bar{\mathcal{W}}$ by imposing upper bounds on the auxiliary variables u and v.

N	T	p/s	Basic			With repositioning capacity		
			MVP	ELDR-BSZ	ELDR	MVP	ELDR-BSZ	ELDR
2	3	0.41/0.32	10.42%	30.85%	34.33%	9.37%	20.85%	30.82%
	4		7.76%	32.04%	33.68%	6.82%	21.90%	32.15%
	5		4.53%	10.35%	13.38%	4.46%	8.25%	12.54%
	3	0.41/0.50	5.83%	19.39%	21.44%	4.89%	12.95%	21.51%
	4		5.31%	18.00%	20.74%	4.81%	13.99%	20.18%
	5		6.55%	15.43%	18.54%	6.20%	11.21%	17.87%
3	3	0.41/0.32	7.56%	16.53%	17.87%	7.03%	12.77%	17.91%
	4		6.83%	20.80%	21.94%	6.04%	13.95%	21.47%
	5		6.29%	13.22%	15.35%	5.88%	11.37%	14.65%
	3	0.41/0.50	4.97%	16.66%	17.69%	4.77%	11.82%	17.72%
	4		4.57%	11.82%	11.93%	3.93%	8.96%	13.19%
	5		5.71%	11.09%	12.16%	4.92%	8.10%	12.32%
4	3	0.41/0.32	6.29%	12.47%	14.38%	5.95%	11.23%	12.59%
	4		7.50%	29.15%	30.51%	6.54%	22.31%	29.79%
	5		5.39%	15.14%	16.54%	4.88%	11.78%	17.01%
	3	0.41/0.50	3.77%	16.28%	17.41%	3.05%	13.16%	15.85%
	4		4.08%	19.05%	19.77%	3.61%	16.37%	19.26%
	5		3.09%	13.75%	16.09%	2.65%	12.46%	15.38%
5	3	0.41/0.32	4.55%	14.19%	15.12%	4.07%	11.84%	14.81%
	4		4.59%	15.18%	17.37%	4.13%	12.88%	16.90%
	5		5.67%	13.59%	15.04%	5.52%	11.99%	15.77%
	3	0.41/0.50	3.64%	13.18%	14.65%	3.26%	10.55%	14.41%
	4		3.94%	14.69%	15.71%	3.47%	11.44%	15.28%
	5		2.95%	13.62%	15.10%	2.61%	9.34%	14.87%

Table 5 **Average VoR of MVP, ELDR-BSZ and ELDR with and without repositioning capacity**

We conduct experiments on both the basic case without repositioning capacity and the case with repositioning capacity R_t set to 15% of the total fleet size. Table 5 presents the average VoR from both ELDR and MVP. Under all instances, our proposed ELDR outperforms ELDR-BSZ. We note that ELDR-BSZ perform closely to ELDR, when there is no repositioning capacity constraint. With the presence of repositioning capacity, however, the performance of ELDR-BSZ is only slightly better than MVP in some cases. Meanwhile, the improvement of our proposed ELDR over ELDR-BSZ is much more significant. The reason is that, when implementing ELDR-BSZ for the case with repositioning capacity, the coefficients for auxiliary variables u and v will be forced to zero in order to satisfy the repositioning capacity constraint. Consequently, the information on the higher moments and cross moments of demands is not used by ELDR-BSZ and the advantage of ELDR is

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lost. By introducing upper bounds on u and v, our ELDR is capable of incorporating such distributional information, even with repositioning capacity constraints.

6. Conclusion

In this paper, we study the fleet repositioning problem for free-float vehicle sharing systems. Specifically, a stochastic dynamic program is developed and we show that the reposition up-to and down-to policy is optimal for a 2-region system. To deal with multi-region systems as well as incorporate demand temporal dependence, we propose two DRO models: the "myopic" two-stage model and the multi-stage ELDR model. Theoretically, we establish the optimality of the ELDR approach in solving single-period DRO problem.

Numerically, we show that in a 2-region system ELDR performs closely to the optimal solution obtained from solving the dynamic program. In a more general multi-region setting, our case study of car2go demonstrates that the ELDR model scales very well and its solution outperforms the "myopic" and MVP. We conclude that ELDR is able to provide sound adaptive repositioning decisions that bring significant value of repositioning. Close investigation between the repositioning and demand patterns suggests that the firm should relocate vehicles to high-demand regions in the early periods and tune down its repositioning activities in later periods. In the extensions with spatial-temporal correlations information and repositioning capacity constraints, we discuss the practical considerations in implementing the ELDR approach, such as the construction of ambiguity set.

While our development of the optimization framework and practical solutions is based on the free-float vehicle sharing operations, the ELDR model can be directly extended to station-based vehicle sharing systems, by adding parking capacity constraints. Furthermore, instead of centrally managing the fleet by the firm, it would be interesting to consider using pricing as an instrument to incentivize customers to move vehicles. For example, Mobike designates some "bonus" bikes that offer cash gifts to customers who ride them with gift sizes depending on the origins and destinations. Thus, a potential further research direction is to develop an optimal pricing scheme for fleet repositioning.

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Online Supplement: Robust Repositioning for Vehicle Sharing

Appendix A: Proofs

A.1. Proof of Lemma 1

Let $f(\mathbf{w_t}) = \sum_{i\in[N]} \bar{p}_{it}(d_{it} - w_{it}) + V_{t+1}\left(\mathbf{x_{t+1}}, \mathbf{d_{[t]}}\right)$ for given $\mathbf{d_{[t]}}$. For the first part, we show that under the condition $\bar{p}_{it} \geq \sum_{j\neq i} s_{ji(t+1)}\alpha_{ijt}$, the solution $w^*_{it} = d_{it} \wedge \left(x_{it} + \sum_{j\in[N]} r_{jit} - \sum_{j\in[N]} r_{ijt}\right)$ is optimal for the minimization problem in (2), which then confirms that $J_t(\mathbf{x_t}, \mathbf{r_t}, \mathbf{d_{[t]}}) = f(\mathbf{w_t}^*)$. Suppose, on the contrary, there exists an optimal solution $\mathbf{w'_t}$ to (2) with $w'_{it} < d_{it} \wedge \left(x_{it} + \sum_{j\in[N]} r_{jit} - \sum_{j\in[N]} r_{ijt}\right)$ for some $i \in [N]$. We denote the state in $t+1$ under $\mathbf{w'_t}$ as $\mathbf{x'_{t+1}}$. Let $\epsilon = d_{it} \wedge \left(x_{it} + \sum_{j\in[N]} r_{jit} - \sum_{j\in[N]} r_{ijt}\right) - w'_{it} > 0$, then the solution constructed by $w_{it} = w'_{it} + \epsilon$ and $w_{jt} = w'_{jt}$ for any $j \neq i$ is still a feasible solution to (2) with a cost $f(\mathbf{w_t}) = \sum_{i\in[N]} \bar{p}_{it}(d_{it} - w_{it}) + V_{t+1}\left(\mathbf{x_{t+1}}, \mathbf{d_{[t]}}\right) = \sum_{i\in[N]} \bar{p}_{it}(d_{it} - w'_{it}) - \bar{p}_{it}\epsilon + V_{t+1}\left(\mathbf{x_{t+1}}, \mathbf{d_{[t]}}\right)$. Observe here that $x_{i(t+1)} = x'_{i(t+1)} - \epsilon + \alpha_{iit}\epsilon$ and $x_{j(t+1)} = x'_{j(t+1)} + \alpha_{ijt}\epsilon$, $\forall j \neq i$. In period $t+1$ at state $\mathbf{x_{t+1}}$, it is always feasible to first reach the state $\mathbf{x'_{t+1}}$ by letting $r_{ji(t+1)} = \alpha_{ijt}\epsilon$ for $j \neq i$, and $r_{jk(t+1)} = 0$ for any $j \in [N], k \neq i$, at a cost $\sum_{j\neq i} s_{ji(t+1)}\alpha_{ijt}\epsilon$. After reaching to state $\mathbf{x'_{t+1}}$, one can reposition again according to the optimal policy at state $\mathbf{x'_{t+1}}$, which yields an optimal cost-to-go $V_{t+1}\left(\mathbf{x'_{t+1}}, \mathbf{d_{[t]}}\right)$. This two-step repositioning, however, may not be optimal at state $\mathbf{x_{t+1}}$. Thus, we must have $V_{t+1}\left(\mathbf{x_{t+1}}, \mathbf{d_{[t]}}\right) \leq \sum_{j\neq i} s_{ji(t+1)}\alpha_{ijt}\epsilon + V_{t+1}\left(\mathbf{x'_{t+1}}, \mathbf{d_{[t]}}\right)$. Consequently,

$$\begin{aligned} f(\mathbf{w_t}) &\leq \sum_{i\in[N]} \bar{p}_{it}(d_{it} - w'_{it}) + V_{t+1}\left(\mathbf{x'_{t+1}}, \mathbf{d_{[t]}}\right) - \bar{p}_{it}\epsilon + \sum_{j\neq i} s_{ji(t+1)}\alpha_{ijt}\epsilon \\ &= f(\mathbf{w'_t}) + \sum_{j\neq i} s_{ji(t+1)}\alpha_{ijt}\epsilon - \bar{p}_{it}\epsilon \\ &\leq f(\mathbf{w'_t}), \end{aligned}$$

where in the last inequality we have used the condition that $\bar{p}_{it} \geq \sum_{j\neq i} s_{ji(t+1)}\alpha_{ijt}$. Hence, $\mathbf{w_t}$ must also be optimal. This establishes that $w^*_{it} = d_{it} \wedge \left(x_{it} + \sum_{j\in[N]} r_{jit} - \sum_{j\in[N]} r_{ijt}\right)$ must be optimal.

For the second part, clearly $V_{T+1}(\mathbf{x_{T+1}})$ is convex. Suppose $V_{t+1}\left(\mathbf{x_{t+1}}, \mathbf{d_{[t]}}\right)$ is convex in $\mathbf{x_{t+1}}$ for $t \leq T$. The objective function in (2) is then jointly convex in $\mathbf{x_t}, \mathbf{r_t}, \mathbf{w_t}$ and the constraint set is a convex set. As a result, for any $\mathbf{d_{[t]}}$, $J_t\left(\mathbf{x_t}, \mathbf{r_t}, \mathbf{d_{[t]}}\right)$ is convex in $\mathbf{x_t}, \mathbf{r_t}$ (e.g., Proposition 2.2.15 in Simchi-Levi et al. 2005), and hence $\mathbb{E}_{\mathbb{P}}[J_t(\mathbf{x_t}, \mathbf{r_t}, \mathbf{d_{[t]}})]$ is also jointly convex in $\mathbf{x_t}, \mathbf{r_t}$ for any given $\mathbf{d_{[t-1]}}$. Thus, the objective function in (1) is jointly convex in $\mathbf{x_t}, \mathbf{r_t}$. As the constraint set in (1) is also convex, $V_t\left(\mathbf{x_t}, \mathbf{d_{[t-1]}}\right)$ is convex in $\mathbf{x_t}$. □

A.2. Proof of Proposition 1

For a 2-region system, the stochastic dynamic program (1) can then be simplified as:

$$V_t(x_t) = \min_{x_t - C \leq r_t \leq x_t} \left\{ s_{12t} r_t^+ + s_{21t} r_t^- + \mathbb{E}_{\mathbb{P}}[J_t(y_t, \mathbf{d_t})] \right\}$$

where

$$\begin{aligned} J_t(y_t, \mathbf{d_t}) = &\min_{w_{1t}, w_{2t}} \left\{ \bar{p}_{1t}(d_{1t} - w_{1t}) + \bar{p}_{2t}(d_{2t} - w_{2t}) + V_{t+1}(x_{t+1}) \right\}, \\ \text{s.t.} \quad & x_{t+1} = y_t - \alpha_{12t} w_{1t} + \alpha_{21t} w_{2t}, \\ & w_{1t} \leq y_t \wedge d_{1t}, \\ & w_{2t} \leq (C - y_t) \wedge d_{2t}, \end{aligned}$$

and the terminal cost $V_{T+1}(x_{T+1}) = 0$.

To prove Proposition 1, we show the lemma below with the index t dropped for the ease of presentation.

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Lemma 4. *If $y = x - r$ and the function $F(\cdot)$ is convex, then there exist $\underline{x}$ and $\overline{x}$ such that the optimal solution to $\min_{x-C\le r\le x}\{s_{12}r^+ + s_{21}r^- + F(y)\}$, denoted as $r^*(x)$, is given by*

$$r^*(x) = \begin{cases} x - \underline{x}, & x \in [0, \underline{x}), \\ 0, & x \in [\underline{x}, \overline{x}], \\ x - \overline{x}, & x \in (\overline{x}, C], \end{cases} \quad \text{and } y^*(x) = \begin{cases} \underline{x}, & x \in [0, \underline{x}), \\ x, & x \in [\underline{x}, \overline{x}], \\ \overline{x}, & x \in (\overline{x}, C]. \end{cases}$$

where $\underline{x}$ and $\overline{x}$ are provided by the following two convex programs

$$\underline{x} = \arg\min_{0\le y\le C}\{s_{21}y + F(y)\}, \quad \overline{x} = \arg\min_{0\le y\le C}\{-s_{12}y + F(y)\}.$$

Proof of Lemma 4: Since $F(y)$ is convex, $F(x-r)$ is submodular in x and r (see Theorem 2.3.6 in Simchi-Levi et al. 2005). Moreover, we see that $[x - C, x]$ is an increasing set in x. By Theorem 2.3.7 in Simchi-Levi et al. (2005), $r^*(x)$ is increasing in x. Note that $r^*(0) \le 0$ and $r^*(C) \ge 0$. Define $\underline{x} = \inf\{x : r^*(x) \ge 0\}$ and $\overline{x} = \sup\{x : r^*(x) \le 0\}$. By monotonicity of $r^*(x)$, it holds $0 \le \underline{x} \le \overline{x} \le C$. By definition, when $x \in [0, \underline{x})$, $r^*(x) < 0$ and $y^*(x) > x$; when $x \in (\overline{x}, C]$, $r^*(x) > 0$ and $y^*(x) < x$; and when $x \in [\underline{x}, \overline{x}]$, $r^*(x) = 0$ and $y^*(x) = x$.

Next, we show that when $x \in [0, \underline{x})$, $y^*(x) = \underline{x}$ and hence $r^*(x) = \underline{x} - x$. Note that when $x \in [0, \underline{x})$, the problem $\min_{x-C\le r\le x}\{s_{12}r^+ + s_{21}r^- + F(y)\}$ is equivalent to $\min_{\substack{0\le y\le C\\ y\ge x}} s_{12}(x-y)^+ + s_{21}(x-y)^- + F(y)$, where we have added the constraint $y \ge x$ and this will not affect the optimality since we know $y^*(x) > x$ for $x \in [0, \underline{x})$. Thus, we can further reduce the problem to

$$\min_{0\le y\le C} s_{21}(y - x) + F(y), \tag{9}$$

where under the dummy constraint $y \ge x$, $(x-y)^+ = 0$ and $(x-y)^- = y - x$. Observe that the solution to (9) does not depend on x; we denote this solution as $\underline{y}$. That is, $y^*(x) = \underline{y}$ when $x \in [0, \underline{x})$. However, we also know that $y^*(x) = x$, when $x \in [\underline{x}, \overline{x}]$. Therefore, by continuity of $y^*(x)$, we must have $\lim_{x\uparrow\underline{x}} y^*(x) = \underline{y} = \underline{x} = y^*(\underline{x})$. The other part when $x \in (\overline{x}, C]$ can be shown similarly. This completes the proof of Lemma 4. □

By Lemma 1, $V_t(x_t)$ is convex in x_t for any $t \in [T]$. As a result, $\mathbb{E}_{\mathbb{P}}[J_t(y, \mathbf{d_t})]$ is convex in y. Proposition 1 then follows directly from Lemma 4 by letting $F(y) = \mathbb{E}_{\mathbb{P}}[J_t(y, \mathbf{d_t})]$. □

A.3. Proof of Corollary 1

From Proposition 1, we know that $\underline{x}_t = y^*(s_{21t}) = \arg\min_{0\le y\le C}\{s_{21t}y + \mathbb{E}_{\mathbb{P}}[J_t(y, \mathbf{d_t})]\}$ and $\overline{x}_t = y^*(s_{12t}) = \arg\min_{0\le y\le C}\{-s_{12t}y + \mathbb{E}_{\mathbb{P}}[J_t(y, \mathbf{d_t})]\}$. Since $s_{21t}y + \mathbb{E}_{\mathbb{P}}[J_t(y, \mathbf{d_t})]$ is supermodular in s_{21t} and y (or submodular in $-s_{21t}$ and y), $\underline{x}_t = y^*(s_{21t})$ is decreasing in s_{21t}. Similarly, $-s_{12t}y + \mathbb{E}_{\mathbb{P}}[J_t(y, \mathbf{d_t})]$ is submodular in s_{12t} and y, thus $\bar{x}_t = y^*(s_{12t})$ is increasing in s_{12t}. □

A.4. Proof of Corollary 2

When $T = 1$, the optimization problem becomes $\min_{x-C\le r\le x}\{s_{12}r^+ + s_{21}r^- + \bar{p}_1\mathbb{E}_{\mathbb{P}}[(d_1 - y)^+] + \bar{p}_2\mathbb{E}_{\mathbb{P}}[(d_2 - C + y)^+]\}$. One can solve the thresholds $\underline{x}$ and $\overline{x}$ as the following two newsvendor type problems: $\underline{x} = \arg\min_{0\le y\le C}\{s_{21}y + \bar{p}_1\mathbb{E}_{\mathbb{P}}[(d_1 - y)^+] + \bar{p}_2\mathbb{E}_{\mathbb{P}}[(d_2 - C + y)^+]\}$ and $\overline{x} = \arg\min_{0\le y\le C}\{-s_{12}y + \bar{p}_1\mathbb{E}_{\mathbb{P}}[(d_1 - y)^+] + \bar{p}_2\mathbb{E}_{\mathbb{P}}[(d_2 - C + y)^+]\}$. Note that by the first-order condition, $\underline{x}_0$ and $\overline{x}_0$ are respectively the unconstrained optimal solution to the above two problems. Since the above two objective functions are convex in y, we must have $\underline{x} = \underline{x}_0^+ \wedge C$ and $\overline{x} = \overline{x}_0^+ \wedge C$.

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A.5. Proof of Proposition 2

This proof follows similar procedures in Bertsimas et al. (2018). First, we show that $\mathbb{F} \subseteq \Pi_{\mathbf{d}}\mathbb{G}$. Suppose $\mathbb{P} \in \mathbb{F}$. We have $\mathbb{P}\left(\left(\mathbf{d}, \{(d_{it}-\mu_{it})^2, \forall i \in [N], t \in [T]\}, \left\{\left(\sum_{l=k}^{t} \mathbf{1}'(\mathbf{d}_l - \boldsymbol{\mu}_l)\right)^2, \forall k, t \in [T], k \leq t\right\} \in \bar{\mathcal{W}}\right)\right) = 1$. We can construct a probability distribution $\mathbb{Q} \in \mathcal{P}_0\left(\mathbb{R}^{NT} \times \mathbb{R}^{NT} \times \mathbb{R}^{\frac{T(T+1)}{2}}\right)$ for $(\mathbf{d}, \mathbf{u}, \mathbf{v})$ by letting $\mathbb{P} = \Pi_{\mathbf{d}}\mathbb{Q}$, $u_{it} = (d_{it} - \mu_{it})^2$ for all $i \in [N], t \in [T]$ and $v_{kt} = \left(\sum_{l=k}^{t} \mathbf{1}'(\mathbf{d}_l - \boldsymbol{\mu}_l)\right)^2$ for all $k, t \in [T], k \leq t$ $\mathbb{P}$-a.s. One can verify that $u_{it} \leq \bar{u}_{it}$ and $v_{kt} \leq \bar{v}_{kt}$ also hold, because $\underline{\mathbf{d}} \leq \mathbf{d} \leq \bar{\mathbf{d}}$. Therefore, $\mathbb{Q}\left((\mathbf{d}, \mathbf{u}, \mathbf{v}) \in \bar{\mathcal{W}}\right) = 1$. Moreover, since $\mathbb{E}_{\mathbb{Q}}(u_{it}) = \mathbb{E}_{\mathbb{P}}((d_{it} - \mu_{it})^2) \leq \sigma_{it}^2$ and $\mathbb{E}_{\mathbb{Q}}(v_{kt}) = \mathbb{E}_{\mathbb{P}}\left(\left(\sum_{l=k}^{t} \mathbf{1}'(\mathbf{d}_l - \boldsymbol{\mu}_l)\right)^2\right) \leq \gamma_{kt}^2$, we have $\mathbb{F} \subseteq \Pi_{\mathbf{d}}\mathbb{G}$.

Next, we show that $\Pi_{\mathbf{d}}\mathbb{G} \subseteq \mathbb{F}$. For any $\mathbb{Q} \in \mathbb{G}$ and $\mathbb{P}$ is the marginal distribution of $\mathbf{d}$ under any $\mathbb{Q}$, i.e., $\mathbb{P} \in \Pi_{\mathbf{d}}\mathbb{G}$, we have $\mathbb{E}_{\mathbb{P}}(\mathbf{d}) = \mathbb{E}_{\mathbb{Q}}(\mathbf{d}) = \boldsymbol{\mu}$. Also, because $\mathbb{Q}\left((\mathbf{d}, \mathbf{u}, \mathbf{v}) \in \bar{\mathcal{W}}\right) = 1$, we have $\mathbb{Q}\left(\mathbf{d} \in (\underline{\mathbf{d}}, \bar{\mathbf{d}})\right) = 1$, $\mathbb{Q}\left((d_{it} - \mu_{it})^2 \leq u_{it}\right) = 1$ for all $i \in [N], t \in [T]$ and $\mathbb{Q}\left(\left(\sum_{l=k}^{t} \mathbf{1}'(\mathbf{d}_l - \boldsymbol{\mu}_l)\right)^2 \leq v_{kt}\right) = 1$ for all $k, t \in [T], k \leq t$. Hence, we have $\mathbb{P}\left(\mathbf{d} \in (\underline{\mathbf{d}}, \bar{\mathbf{d}})\right) = 1$, $\mathbb{E}_{\mathbb{P}}\left((d_{it} - \mu_{it})^2\right) = \mathbb{E}_{\mathbb{Q}}\left((d_{it} - \mu_{it})^2\right) \leq \mathbb{E}_{\mathbb{Q}}(u_{it}) \leq \sigma_{it}^2$ and $\mathbb{E}_{\mathbb{P}}\left(\left(\sum_{l=k}^{t} \mathbf{1}'(\mathbf{d}_l - \boldsymbol{\mu}_l)\right)^2\right) = \mathbb{E}_{\mathbb{Q}}\left(\left(\sum_{l=k}^{t} \mathbf{1}'(\mathbf{d}_l - \boldsymbol{\mu}_l)\right)^2\right) \leq \mathbb{E}_{\mathbb{Q}}(v_{kt}) \leq \gamma_{kt}^2$. Thus, $\mathbf{P} \in \mathbb{F}$. Consequently, we conclude that $\Pi_{\mathbf{d}}\mathbb{G} \subseteq \mathbb{F}$. □

A.6. Proof of Lemma 2

Given the first stage decision $\mathbf{r}$, we derive $\sup_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}}\left[\bar{p}_i\left(d_i - w_i(\mathbf{d})\right)\right]$. By Proposition 2, we have $\sup_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}}\left[\bar{p}_i\left(d_i - w_i(\mathbf{d})\right)\right] = \sup_{\mathbb{Q} \in \mathbb{G}} \mathbb{E}_{\mathbb{Q}}\left[\bar{p}_i\left(d_i - w_i(\mathbf{d})\right)\right]$, under ambiguity sets $\mathbb{F}$ and $\mathbb{G}$. The second stage worst-case cost can then be calculated by the following infinite dimensional linear program:

$$
\begin{aligned}
\max_{\mathbb{Q} \in \mathbb{G}} \quad & \int_{(\mathbf{d}, \mathbf{u}, v) \in \bar{\mathcal{W}}} \sum_{i \in [N]} \bar{p}_i\left(d_i - w_i(\mathbf{d})\right) d\mathbb{Q}\left(\mathbf{d}, \mathbf{u}, v\right) \\
\text{s.t.} \quad & \int_{(\mathbf{d}, \mathbf{u}, v) \in \bar{\mathcal{W}}} \mathbf{d} d\mathbb{Q}\left(\mathbf{d}, \mathbf{u}, v\right) = \boldsymbol{\mu} && (\cdots \text{ dual variable } \boldsymbol{\eta} \in \mathbb{R}^N) \\
& \int_{(\mathbf{d}, \mathbf{u}, v) \in \bar{\mathcal{W}}} u_i d\mathbb{Q}\left(\mathbf{d}, \mathbf{u}, v\right) \leq \sigma_i^2, \forall i \in [N] && (\cdots \text{ dual variable } \theta_i \in \mathbb{R}) \\
& \int_{(\mathbf{d}, \mathbf{u}, v) \in \bar{\mathcal{W}}} v d\mathbb{Q}\left(\mathbf{d}, \mathbf{u}, v\right) \leq \gamma^2 && (\cdots \text{ dual variable } \delta \in \mathbb{R}) \\
& \int_{(\mathbf{d}, \mathbf{u}, v) \in \bar{\mathcal{W}}} d\mathbb{Q}\left(\mathbf{d}, \mathbf{u}, v\right) = 1 && (\cdots \text{ dual variable } \lambda \in \mathbb{R})
\end{aligned}
$$

By the strong duality (Bertsimas et al. 2018), we have the dual formulation as

$$
\begin{aligned}
\min_{\boldsymbol{\theta} \geq \mathbf{0}, \delta \geq 0, \lambda, \boldsymbol{\eta}} \quad & \lambda + \boldsymbol{\eta}'\boldsymbol{\mu} + \sum_{i \in [N]} \sigma_i^2 \theta_i + \gamma^2 \delta \\
\text{s.t.} \quad & \lambda + \boldsymbol{\eta}'\boldsymbol{d} + \sum_{i \in [N]} \theta_i u_i + \delta v \geq \sum_{i \in [N]} \bar{p}_i\left(d_i - w_i(\mathbf{d})\right), \forall (\mathbf{d}, \boldsymbol{u}, v) \in \bar{\mathcal{W}}
\end{aligned}
$$

By $w_i(\mathbf{d}) = d_i \wedge \left(x_i + \sum_{j \in [N]} r_{ji} - \sum_{j \in [N]} r_{ij}\right)$, we have $(d_i - w_i(\mathbf{d})) = \left(d_i - x_i - \sum_{j \in [N]} r_{ji} + \sum_{j \in [N]} r_{ij}\right)^+$. Combining the second stage worst-case cost with the first stage problem, we have the formulation (4). □

A.7. Proof of Proposition 3

Let $\mathcal{P}(N)$ be the power set of $[N]$. Since $\left(d_i - x_i - \sum_{j \in [N]} r_{ji} + \sum_{j \in [N]} r_{ij}\right)^+$ takes value either $d_i - x_i - \sum_{j \in [N]} r_{ji} + \sum_{j \in [N]} r_{ij}$ or 0, we are able to write the sum of piecewise linear functions as

$$
\sum_{i \in [N]} \bar{p}_i\left(d_i - x_i - \sum_{j \in [N]} r_{ji} + \sum_{j \in [N]} r_{ij}\right)^+ = \max_{S \in \mathcal{P}(N)} \sum_{i \in S} \bar{p}_i\left(d_i - x_i - \sum_{j \in [N]} r_{ji} + \sum_{j \in [N]} r_{ij}\right).
$$

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Let $y_i = x_i + \sum_{j\in[N]} r_{ji} - \sum_{j\in[N]} r_{ij}$, the first constraint in problem (4) can then be written as $\lambda + \boldsymbol{\eta}'\mathbf{d} + \sum_{i\in[N]} \theta_i u_i + \delta v \geq \max_{S\in\mathcal{P}(N)} \sum_{i\in S} \bar{p}_i (d_i - y_i), \forall(\mathbf{d}, \boldsymbol{u}, v) \in \bar{\mathcal{W}}$. This is also equivalent to

$$\lambda + \boldsymbol{\eta}'\mathbf{d} + \sum_{i\in[N]} \theta_i u_i + \delta v \geq \sum_{i\in S} \bar{p}_i (d_i - y_i), \forall S \in \mathcal{P}(N), \forall(\mathbf{d}, \boldsymbol{u}, v) \in \bar{\mathcal{W}} \tag{10}$$

We reorganize the terms in the above constraint into: $\lambda + \sum_{i\in S} \bar{p}_i y_i \geq \max_{(\mathbf{d},\boldsymbol{u},v)\in\bar{\mathcal{W}}} \left\{ (\bar{\mathbf{p}}(S) - \boldsymbol{\eta})' \mathbf{d} - \sum_{i\in[N]} \theta_i u_i - \delta v \right\}, \forall S \in \mathcal{P}(N)$. The subproblem on the right-hand-side (RHS) is a SOCP. To see this, we can equivalently rewrite $\bar{\mathcal{W}}$ into a set of second-order conic constraints:

$$\bar{\mathcal{W}} \equiv \left\{ (\mathbf{d}, \mathbf{u}, \mathbf{v}) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \;\middle|\; \begin{array}{l} \underline{\mathbf{d}} \leq \mathbf{d} \leq \bar{\mathbf{d}} \\ \sqrt{(d_i - \mu_i)^2 + \left(\frac{u_i - 1}{2}\right)^2} \leq \frac{u_i+1}{2}, \ \forall i \in [N] \\ \sqrt{(\mathbf{1}'(\mathbf{d} - \boldsymbol{\mu}))^2 + \left(\frac{v-1}{2}\right)^2} \leq \frac{v+1}{2} \end{array} \right\}.$$

Note that the subproblem $\max_{(\boldsymbol{d},\boldsymbol{u},\boldsymbol{v})\in\bar{\mathcal{W}}} \left\{ (\bar{\mathbf{p}}(S) - \boldsymbol{\eta})' \boldsymbol{d} - \sum_{i\in[N]} \theta_i u_i - \delta v \right\}$ is bounded and there exists an interior point in $\bar{\mathcal{W}}$. Let $\bar{\boldsymbol{\beta}}(S) \in \mathbb{R}^N$, $\underline{\boldsymbol{\beta}}(S) \in \mathbb{R}^N$, $\boldsymbol{\beta}(S) \in \mathbb{R}^N$, $\beta_0(S) \in \mathbb{R}$, $\boldsymbol{\rho}_i(S) \in \mathbb{R}^2$ and $\boldsymbol{\rho}_0(S) \in \mathbb{R}^2$ be the corresponding dual variables. By strong duality, the equivalent dual problem is also a SOCP:

$$\begin{aligned} \min_{\bar{\boldsymbol{\beta}}(S), \underline{\boldsymbol{\beta}}(S) \geq \mathbf{0}} \quad & \bar{\boldsymbol{\beta}}(S)'\bar{\mathbf{d}} - \underline{\boldsymbol{\beta}}(S)'\underline{\mathbf{d}} + \frac{1}{2}\mathbf{1}'\boldsymbol{\beta}(S) + \frac{1}{2}\beta_0(S) - \sum_{i\in[N]} \boldsymbol{\rho}_i(S)'\mathbf{b}_i - \boldsymbol{\rho}_0(S)\mathbf{b}_0 \\ \text{s.t.} \quad & \begin{pmatrix} \boldsymbol{\eta} - \bar{\mathbf{p}}(S) \\ \boldsymbol{\theta} \\ \delta \end{pmatrix} = \begin{pmatrix} \underline{\boldsymbol{\beta}}(S) - \bar{\boldsymbol{\beta}}(S) \\ \mathbf{0} \\ 0 \end{pmatrix} + \sum_{i\in[N]} \left(\mathbf{A}_i'\boldsymbol{\rho}_i(S) + \beta_i(S)\mathbf{c}_i \right) + \mathbf{A}_0'\boldsymbol{\rho}_0(S) + \beta_0\mathbf{c}_0 \\ & \|\boldsymbol{\rho}_i(S)\|_2 \leq \beta_i(S), \forall i \in [N] \\ & \|\boldsymbol{\rho}_0(S)\|_2 \leq \beta_0(S) \end{aligned}$$

where $\mathbf{A}_0 = \begin{bmatrix} \mathbf{1}' & \mathbf{0}' & 0 \\ \mathbf{0}' & \mathbf{0}' & \frac{1}{2} \end{bmatrix}$, $\mathbf{b}_0 = \begin{pmatrix} \mathbf{1}'\boldsymbol{\mu} \\ \frac{1}{2} \end{pmatrix}$, $\mathbf{c}_0 = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \frac{1}{2} \end{pmatrix}$, $\mathbf{A}_i = \begin{bmatrix} \mathbf{e}_i' & \mathbf{0}' & 0 \\ \mathbf{0} & \frac{1}{2}\mathbf{e}_i' & 0 \end{bmatrix}$, $\mathbf{b}_i = \begin{pmatrix} \mu_i \\ \frac{1}{2} \end{pmatrix}$, $\mathbf{c}_i = \begin{pmatrix} \mathbf{0} \\ \frac{1}{2}\mathbf{e}_i \\ 0 \end{pmatrix}$, $\forall i \in [N]$.

We are now able to replace the maximization problem in constraint (10) with the dual problem. Consequently, constraint (10) is equivalently saying that there exists a feasible solution $\left(\bar{\boldsymbol{\beta}}(S), \underline{\boldsymbol{\beta}}(S), \boldsymbol{\beta}(S), \beta_0(S), \boldsymbol{\rho}_i(S), \boldsymbol{\rho}_0(S)\right)$ satisfying the following constraints for all $S \in \mathcal{P}(N)$:

$$\begin{aligned} & \lambda + \sum_{i\in S} \bar{p}_i y_i \geq \bar{\boldsymbol{\beta}}(S)'\bar{\mathbf{d}} - \underline{\boldsymbol{\beta}}(S)'\underline{\mathbf{d}} + \frac{1}{2}\mathbf{1}'\boldsymbol{\beta}(S) + \frac{1}{2}\beta_0(S) - \sum_{i\in[N]} \boldsymbol{\rho}_i(S)'\mathbf{b}_i - \boldsymbol{\rho}_0(S)\mathbf{b}_0 \\ & \begin{pmatrix} \boldsymbol{\eta} - \bar{\mathbf{p}}(S) \\ \boldsymbol{\theta} \\ \delta \end{pmatrix} = \begin{pmatrix} \underline{\boldsymbol{\beta}}(S) - \bar{\boldsymbol{\beta}}(S) \\ \mathbf{0} \\ 0 \end{pmatrix} + \sum_{i\in[N]} \left(\mathbf{A}_i'\boldsymbol{\rho}_i(S) + \beta_i(S)\mathbf{c}_i \right) + \mathbf{A}_0'\boldsymbol{\rho}_0(S) + \beta_0\mathbf{c}_0 \\ & \|\boldsymbol{\rho}_i(S)\|_2 \leq \beta_i(S), \forall i \in [N] \\ & \|\boldsymbol{\rho}_0(S)\|_2 \leq \beta_0(S) \\ & \bar{\boldsymbol{\beta}}(S), \underline{\boldsymbol{\beta}}(S) \geq \mathbf{0} \end{aligned}$$

By replacing the first constraint in problem (4) with the above, we have the SOCP in the proposition. □

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A.8. Proof of Proposition 4

Before proving Proposition 4, we first introduce several notations. Let $\hat{w}_i(\mathbf{d}) = d_i - w_i(\mathbf{d})$, $\hat{w}_i(\mathbf{d},\mathbf{u},v) = d_i - w_i(\mathbf{d},\mathbf{u},v)$, $y_i = x_i + \sum_{j\in[N]} r_{ji} - \sum_{j\in[N]} r_{ij}$,

$$\begin{aligned}\beta(\mathbf{y}) = \min_{\hat{w}_i(\cdot)} \sup_{\mathbb{P}\in\mathbb{F}} & \mathbb{E}_{\mathbb{P}}\left[\sum_{i\in[N]} \bar{p}_i \hat{w}_i(\mathbf{d})\right] \\ \text{s.t.} \quad & \hat{w}_i(\mathbf{d}) \geq (d_i - y_i)^+, \forall \mathbf{d}\in\mathcal{W}, i\in[N].\end{aligned} \tag{11}$$

and

$$\begin{aligned}\beta^{\mathrm{ELDR}}(\mathbf{y}) = \min_{\hat{w}_i(\cdot)} \sup_{\mathbb{Q}\in\mathbb{G}} & \mathbb{E}_{\mathbb{Q}}\left[\sum_{i\in[N]} \bar{p}_i \hat{w}_i(\mathbf{d},\mathbf{u},v)\right] \\ \text{s.t.} \quad & \hat{w}_i(\mathbf{d},\mathbf{u},v) \geq (d_i - y_i)^+, \forall (\mathbf{d},\mathbf{u},v)\in\bar{\mathcal{W}}, i\in[N], \\ & \hat{\mathbf{w}}(\cdot)\in\bar{\mathcal{L}}^N(\mathbf{d},\mathbf{u},v).\end{aligned} \tag{12}$$

Here $\hat{w}_i(\mathbf{d})$ in $\beta(\mathbf{y})$ refers to any decision rule and $w_i(\mathbf{d},\mathbf{u},v)$ in $\beta^{\mathrm{ELDR}}(\mathbf{y})$ is restricted to ELDR. Then,

$$Z^* = \min_{\sum_{j\in[N]} r_{ij}\leq x_i, r_{ij}\geq 0} \left\{\sum_{i,j\in[N]} s_{ij}r_{ij} + \beta(\mathbf{y})\right\}, \quad Z^{\mathrm{ELDR}} = \min_{\sum_{j\in[N]} r_{ij}\leq x_i, r_{ij}\geq 0} \left\{\sum_{i,j\in[N]} s_{ij}r_{ij} + \beta^{\mathrm{ELDR}}(\mathbf{y})\right\}.$$

In addition, for $i\in[N]$, let

$$\begin{aligned}\beta_i(y_i) = \min_{\hat{w}_i(\cdot)} \sup_{\mathbb{P}_i\in\mathbb{F}_i} & \mathbb{E}_{\mathbb{P}_i}\left[\bar{p}_i \hat{w}_i(d_i)\right] \\ \text{s.t.} \quad & \hat{w}_i(d_i) \geq (d_i - y_i)^+, \forall d_i\in\mathcal{W}_i,\end{aligned}$$

$$\text{where} \quad \mathbb{F}_i = \left\{ \mathbb{P}\in\mathcal{P}_0(\mathbb{R}) \;\middle|\; \begin{array}{l} \mathbb{E}_{\mathbb{P}}(d_i) = \mu_i \\ \mathbb{E}_{\mathbb{P}}\left((d_i-\mu_i)^2\right) \leq \sigma_i^2, \\ \mathbb{P}\left(d_i \in (\underline{d}_i, \bar{d}_i)\right) = 1 \end{array} \right\}, \quad \text{and} \quad \mathcal{W}_i = [\underline{d}_i, \bar{d}_i], \quad \text{and let}$$

$$\begin{aligned}\beta_i^{\mathrm{ELDR}}(y_i) = \min_{\hat{w}_i(\cdot)} \sup_{\mathbb{Q}_i\in\mathbb{G}_i} & \mathbb{E}_{\mathbb{Q}_i}\left[\bar{p}_i \hat{w}_i(d_i,u_i)\right] \\ \text{s.t.} \quad & \hat{w}_i(d_i,u_i) \geq (d_i - y_i)^+, \forall (d_i,u_i)\in\bar{\mathcal{W}}_i, \\ & \hat{w}_i(\cdot)\in\bar{\mathcal{L}}(d_i,u_i),\end{aligned}$$

$$\text{where} \quad \mathbb{G}_i = \left\{ \mathbb{Q}\in\mathcal{P}_0(\mathbb{R}\times\mathbb{R}) \;\middle|\; \begin{array}{l} \mathbb{E}_{\mathbb{Q}}(d_i) = \mu_i \\ \mathbb{E}_{\mathbb{Q}}(u_i) \leq \sigma_i^2, \\ \mathbb{Q}\left((d_i,u_i)\in\bar{\mathcal{W}}_i\right) = 1 \end{array} \right\}, \quad \text{with} \quad \bar{\mathcal{W}}_i = \left\{(d,u) \middle|\; \underline{d}_i \leq d_i \leq \bar{d}_i, (d_i-\mu_i)^2 \leq u_i \leq \bar{u}_i\right\}.$$

To show that $Z^{\mathrm{ELDR}} = Z^*$, it is then sufficient to show that $\beta^{\mathrm{ELDR}}(\mathbf{y}) = \beta(\mathbf{y})$ for any $\mathbf{y}$. We establish this via the following two lemmas.

LEMMA 5. *If $\gamma \geq \sqrt{\sum_{i\in[N]} \sigma_i^2}$, then $\beta(\mathbf{y}) = \sum_{i\in[N]} \beta_i(y_i)$ for any $\mathbf{y}$.*

Proof First note that $\beta(\mathbf{y}) = \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left[\sum_{i\in[N]} \bar{p}_i(d_i-y_i)^+\right]$ and $\sum_{i\in[N]} \beta_i(y_i) = \sum_{i\in[N]} \sup_{\mathbb{P}_i\in\mathbb{F}_i} \mathbb{E}_{\mathbb{P}_i}\left[\bar{p}_i(d_i-y_i)^+\right]$. We first show that $\beta(\mathbf{y}) \leq \sum_{i\in[N]} \beta_i(y_i)$. Indeed, for any $\mathbb{P}\in\mathbb{F}$, let $\mathbb{P}_i$ be the marginal distribution for d_i. By the feasibility of $\mathbb{P}$, we have $\mathbb{E}_{\mathbb{P}_i}(d_i) = \mu_i$, $\mathbb{E}_{\mathbb{P}_i}\left((d_i-\mu_i)^2\right) \leq \sigma_i^2$, and $\mathbb{P}_i\left(d_i\in(\underline{d}_i,\bar{d}_i)\right) = 1$. That is, $\mathbb{P}_i\in\mathbb{F}_i$. In addition, $\mathbb{E}_{\mathbb{P}}\left[\sum_{i\in[N]} \bar{p}_i(d_i-y_i)^+\right] = \sum_{i\in[N]} \mathbb{E}_{\mathbb{P}_i}\left[\bar{p}_i(d_i-y_i)^+\right] \leq \sum_{i\in[N]} \sup_{\mathbb{P}_i\in\mathbb{F}_i} \mathbb{E}_{\mathbb{P}_i}\left[\bar{p}_i(d_i-y_i)^+\right]$. This shows that $\beta(\mathbf{y}) \leq \sum_{i\in[N]} \beta_i(y_i)$.

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On the other hand, for any $\mathbb{P}_i \in \mathbb{F}_i$, we construct a joint distribution $\mathbb{P}$ for $(d_1, ..., d_N)$ as $\mathbb{P}(d_1 \le \omega_1, ..., d_N \le \omega_N) = \prod_{i\in[N]} \mathbb{P}_i(d_i \le \omega_i)$, $\forall \omega_i \in [\underline{d}_i, \bar{d}_i]$. In other word, given the marginal distributions of $d_i, i \in [N]$, we can find a joint distribution with the marginal distributions such that $d_i, i \in [N]$ are independent. To verify $\mathbb{P} \in \mathbb{F}$, note that $\mathbb{E}_{\mathbb{P}}(d_i) = \mu_i$, $\mathbb{E}_{\mathbb{P}}\left((d_i - \mu_i)^2\right) \le \sigma_i^2$, and $\mathbb{P}\left(d_i \in (\underline{d}_i, \bar{d}_i)\right) = 1$, by the feasibility of $\mathbb{P}_i$. By the independence of $\mathbb{P}_i$, and the assumption that $\gamma \ge \sqrt{\sum_{i\in[N]} \sigma_i^2}$, we have $\mathbb{E}_{\mathbb{P}}\left((\sum_{i\in[N]}(d_i - \mu_i))^2\right) = \sum_{i\in[N]} \mathbb{E}_{\mathbb{P}}((d_i - \mu_i)^2) = \sum_{i\in[N]} \sigma_i^2 \le \gamma^2$. Hence, $\mathbb{P} \in \mathbb{F}$, and it follows that $\sum_{i\in[N]} \mathbb{E}_{\mathbb{P}_i}[\bar{p}_i(d_i - y_i)^+] = \mathbb{E}_{\mathbb{P}}\left[\sum_{i\in[N]} \bar{p}_i(d_i - y_i)^+\right] \le \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left[\sum_{i\in[N]} \bar{p}_i(d_i - y_i)^+\right]$, i.e., $\sum_{i\in[N]} \beta_i(y_i) \le \beta(\mathbf{y})$. □

LEMMA 6. *For any y_i, $\beta_i(y_i) = \beta_i^{ELDR}(y_i)$.*

Proof Theorem 4 in Bertsimas et al. (2018) shows that ELDR can obtain the optimal objective value for linear DRO problem with only one second stage decision variable. Recall that y_i is the only second stage decision variable in $\beta_i(y_i)$, ELDR can achieve the same optimal objective value. □

We now complete the proof of Proposition 4. Let $\hat{w}_i^*(d_i, u_i) = w_i^0 + w_i^1 d_i + w_i^2 u_i$ be the optimal solution in solving $\beta_i^{\text{ELDR}}(y_i)$. Let $\hat{w}_i^*(\mathbf{d}, \mathbf{u}, v) = w_i^0 + w_i^1 d_i + w_i^2 u_i$. Clearly, $\hat{w}_i^*(\mathbf{d}, \mathbf{u}, v) \in \bar{\mathcal{L}}(\mathbf{d}, \mathbf{u}, v)$ since the linear coefficients corresponding to the variables $d_j, u_j, j \ne i$ and v are simply zero. In addition, for any $i \in [N]$, $\hat{w}_i^*(\mathbf{d}, \mathbf{u}, v) \ge (d_i - y_i)^+, \forall (\mathbf{d}, \mathbf{u}, v) \in \bar{\mathcal{W}}$ by the feasibility of $\hat{w}_i^*(d_i, u_i)$. Hence, $\hat{w}_i^*(\mathbf{d}, \mathbf{u}, v)$ is a feasible solution to (12). It follows that $\beta^{\text{ELDR}}(\mathbf{y}) \le \sup_{\mathbb{Q}\in\mathbb{G}} \mathbb{E}_{\mathbb{Q}}\left[\sum_{i\in[N]} \bar{p}_i \hat{w}_i^*(\mathbf{d}, \mathbf{u}, v)\right]$. For any $\mathbb{Q} \in \mathbb{G}$, let $\mathbb{Q}_i$ be the marginal distribution of (d_i, u_i). It is easy to see that $\mathbb{Q}_i \in \mathbb{G}_i$. It then follows that $\mathbb{E}_{\mathbb{Q}}\left[\sum_{i\in[N]} \bar{p}_i \hat{w}_i^*(\mathbf{d}, \mathbf{u}, v)\right] = \sum_{i\in[N]} \mathbb{E}_{\mathbb{Q}_i}[\bar{p}_i \hat{w}_i^*(d_i, u_i)] \le \sum_{i\in[N]} \sup_{\mathbb{Q}_i\in\mathbb{G}_i} \mathbb{E}_{\mathbb{Q}_i}[\bar{p}_i \hat{w}_i^*(d_i, u_i)] = \sum_{i\in[N]} \beta_i^{\text{ELDR}}(y_i)$. Thus, $\beta^{\text{ELDR}}(\mathbf{y}) \le \sup_{\mathbb{Q}\in\mathbb{G}} \mathbb{E}_{\mathbb{Q}}\left[\sum_{i\in[N]} \bar{p}_i \hat{w}_i^*(\mathbf{d}, \mathbf{u}, v)\right] \le \sum_{i\in[N]} \beta_i^{\text{ELDR}}(y_i)$. By Lemma 5 and Lemma 6, we know that for $\gamma \ge \sqrt{\sum_{i\in[N]} \sigma_i^2}$, we have $\beta(\mathbf{y}) = \sum_{i\in[N]} \beta_i(y_i) = \sum_{i\in[N]} \beta_i^{\text{ELDR}}(y_i)$. Therefore, $\beta^{\text{ELDR}}(\mathbf{y}) \le \beta(\mathbf{y})$. On the other hand, for any $\hat{\mathbf{w}}(\cdot) \in \bar{\mathcal{L}}^N(\mathbf{d}, \mathbf{u}, v)$, it must also be a feasible solution to (11). Hence, $\beta^{\text{ELDR}}(\mathbf{y}) \ge \beta(\mathbf{y})$, which then implies $\beta^{\text{ELDR}}(\mathbf{y}) = \beta(\mathbf{y})$. Therefore, we have $Z^{\text{ELDR}} = Z^*$. □

A.9. Proof of Robust Repositioning Policy in a 2-Region System

Consider a 2-region system with regions 1 and 2. For any period $t \in [T]$, we denote r_{12t} as the vehicle repositioning from region 1 to 2 and r_{21t} from region 2 to 1. Similar to Section 3.1, let x_t be the available vehicles at region 1 at the beginning of period t. The rest of the variables follow from the previous discussions. Our robust model for the 2-region system can be formulated as below.

$$\begin{aligned} \min_{\substack{0\le r_{121}\le x_1 \\ 0\le r_{211}\le C-x_1}} \quad & s_{121}r_{121} + s_{211}r_{211} + F(y_1) \\ \text{s.t.} \quad & y_1 = x_1 - r_{121} + r_{211} \end{aligned} \tag{13}$$

where $F(y_1)$ is directly from formulation (7) by setting $N = 2$.

By Lemma 3, we can write $F(y_1)$ as the optimal value to the following infinite dimensional linear program:

$$\min_{\substack{\boldsymbol{\theta},\boldsymbol{\delta}\ge\mathbf{0},\lambda,\boldsymbol{\eta} \\ \mathbf{x}^0_{t+1},\mathbf{x}^1_{il(t+1)},\mathbf{x}^2_{il(t+1)},\mathbf{x}^3_{kl(t+1)} \\ \mathbf{r}^0_{t+1},\mathbf{r}^1_{il(t+1)},\mathbf{r}^2_{il(t+1)},\mathbf{r}^3_{kl(t+1)} \\ \mathbf{w}^0_t,\mathbf{w}^1_{ilt},\mathbf{w}^2_{ilt},\mathbf{w}^3_{klt}}} \lambda + \boldsymbol{\eta}'\boldsymbol{\mu} + \sum_{\substack{i\in[2] \\ t\in[T]}} \sigma_i^2\theta_i + \sum_{\substack{k,t\in[T] \\ k\le t}} \gamma_{kt}^2\delta_{kt}$$

s.t. Constraints in (7)

$$\lambda + \boldsymbol{\eta}'\boldsymbol{d} + \sum_{\substack{i\in[2] \\ t\in[T]}} \theta_{it}u_{it} + \sum_{\substack{k,t\in[T] \\ k\le t}} \delta_{kt}v_{kt} \ge \sum_{\substack{i\in[2] \\ t\in[T]}} \bar{p}_{it}(d_{it} - w_{it}(\cdot)) + \sum_{\substack{t\in[T-1] \\ i,j\in[2]}} s_{ij(t+1)}r_{ij(t+1)}(\cdot), \forall(\boldsymbol{d},\boldsymbol{u},\boldsymbol{v}) \in \bar{\mathcal{W}}.$$

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All constraints in the above program are linear and hence the constraint set is convex. In addition, the objective is linear, which is then jointly convex in all decision variables and y_1. As a result, $F(y_1)$ is convex in y_1 (see, for example, Proposition 2.2.15 in Simchi-Levi et al. 2005). Using Lemma 4 in the Appendix, we then have the following result on the structure of the robust repositioning policy.

Consider a 2-region system. For any x_1, let r^*_{121}, r^*_{211} be the optimal solution to (13), $r^{\mathrm{R}}_1(x_1) = r^*_{121} - r^*_{211}$ and $y^{\mathrm{R}}_1(x_1) = x_1 - r^{\mathrm{R}}_1(x_1)$, then there exist $\underline{x}_1$ and $\overline{x}_1$ such that

$$r^{\mathrm{R}}_1(x_1) = \begin{cases} x_1 - \underline{x}_1, & x_1 \in [0, \underline{x}_1), \\ 0, & x_1 \in [\underline{x}_1, \overline{x}_1], \\ x_1 - \overline{x}_1, & x \in (\overline{x}_1, C], \end{cases} \quad \text{and} \quad y^{\mathrm{R}}_1(x_1) = \begin{cases} \underline{x}_1, & x \in [0, \underline{x}_1), \\ x_1, & x_1 \in [\underline{x}_1, \overline{x}_1], \\ \overline{x}_1, & x_1 \in (\overline{x}_1, C]. \end{cases}$$

where $\underline{x}_1$ and $\overline{x}_1$ are solutions to the following two convex programs

$$\underline{x}_1 = \arg\min_{0 \le y \le C} \{s_{211} y + F(y)\}, \quad \overline{x}_1 = \arg\min_{0 \le y \le C} \{-s_{121} y + F(y)\}.$$

The result confirms the desirable structural properties of our robust repositioning policy, even without the assumption of demand temporal independence.

Appendix B: Data and Setup for Numerical Studies

B.1. The MVP Formulation

The MVP is solved as the following linear program:

$$\begin{aligned} \min_{r_{ijt} \ge 0} \quad & \sum_{i \in [N]} \left(\sum_{j \in [N]} s_{ijt} r_{ijt} + \bar{p}_{it} (\mu_{it} - w_{it}) \right) \\ \text{s.t.} \quad & x_{i(t+1)} = x_{it} + \sum_{j \in [N]} (\alpha_{jit} w_{jt} + r_{jit}) - \sum_{j \in [N]} (\alpha_{ijt} w_{it} + r_{ijt}), \forall i \in [N], t \in [T] \\ & \sum_{j \in [N]} r_{ijt} \le x_{it}, \forall i \in [N], t \in [T] \\ & w_{it} \le \mu_{it} \wedge \left(x_{it} + \sum_{j \in [N]} r_{jit} - \sum_{j \in [N]} r_{ijt} \right), \forall i \in [N], t \in [T]. \end{aligned}$$

B.2. Parameters for 2-Region System

The trip distribution, lost sales penalty per trip, and repositioning cost per trip are

$$(\alpha_{ijt}) = \begin{bmatrix} 0.87 & 0.13 \\ 0.72 & 0.28 \end{bmatrix}, (p_{ijt}) = \begin{bmatrix} 15.34 & 12.58 \\ 15.05 & 9.60 \end{bmatrix}, (s_{ijt}) = \begin{bmatrix} 13.81 & 11.33 \\ 13.55 & 8.63 \end{bmatrix}, \forall t \in [T].$$

B.3. Sample Data of car2go San Diego

Here we provide 10 lines of sample vehicle status data that record the vehicle ID, address, GPS coordinates and time stamp. We then use the vehicle status data to track the movement of each vehicle and identify trips by checking the changes of locations as well as the trip durations.

B.4. Region Clustering

Based on the inter-region travel intensities, we cluster the zip codes into 5 regions as below:

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Table 6 Sample Data of Vehicle Status in car2go San Diegon

Vehicle ID	Address	Coordinates	TimeGMT+8
6UK E956	India St 2166, 92101	[-117.16968,32.72708,0]	23/3/14 19:41
6UK F014	Collier Ave 3490, 92116	[-117.11775,32.76548,0]	23/3/14 19:41
6RF N765	India St 3209, 92103	[-117.17541,32.73672,0]	23/3/14 19:41
6UK C946	Park Blvd 4113, 92103	[-117.14623,32.75251,0]	23/3/14 19:41
6UK E958	Mission Blvd 3822, 92109	[-117.25317,32.7852,0]	23/3/14 19:41
6UK F000	Camino de la Reina 663, 92108	[-117.15991,32.76641,0]	23/3/14 19:41
6TK Z247	Alvarado Rd 6329, 92120	[-117.06368,32.77833,0]	23/3/14 19:41
6UK E953	Mission Center Ct 7820, 92108	[-117.15555,32.77348,0]	23/3/14 19:41
6RF N728	Columbia St 2688, 92103	[-117.17116,32.73238,0]	23/3/14 19:41
6RF N708	Sunset St 2638, 92110	[-117.19451,32.75549,0]	23/3/14 19:41

Table 7 Region Clustering for car2go San Diego

Region	Zip Codes
1	92101, 92102, 92134, 92132
2	92103, 92104, 92105
3	92108, 92110, 92116
4	92106, 92107
5	92109

B.5. Ambiguity Sets for Section 5.3

We estimate the correlation coefficient ρ_{ikjt} between the demand in region i at time k and the demand in region j at time t. The ambiguity sets $\mathbb{F}_1$ and $\mathbb{F}_2$ are formulated as

$$\mathbb{F}_1 = \left\{ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}^{NT}) \;\middle|\; \begin{array}{ll} \mathbb{E}_{\mathbb{P}}(\mathbf{d}) = \boldsymbol{\mu} & \\ \mathbb{E}_{\mathbb{P}}\left((d_{it} - \mu_{it})^2 \right) \le \sigma_{it}^2, & \forall i \in [N], t \in [T] \\ \mathbb{E}_{\mathbb{P}}\left(\left(\sum_{l=k}^{t} \mathbf{1}'(\mathbf{d}_l - \boldsymbol{\mu}_l) \right)^2 \right) \le \gamma_{kt}^2, & \forall k, t \in [T], k \le t \\ \mathbb{E}_{\mathbb{P}}\left((d_{ik} - \mu_{ik})(d_{jt} - \mu_{jt}) \right) \le \phi_{ikjt}^2, & \forall (i,k,j,t) \in \{(i,k,j,t) | \rho_{ikjt} \ge 0.5\} \\ \mathbb{P}\left(\mathbf{d} \in (\underline{\mathbf{d}}, \bar{\mathbf{d}}) \right) = 1 & \end{array} \right\},$$

and

$$\mathbb{F}_2 = \left\{ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}^{NT}) \;\middle|\; \begin{array}{ll} \mathbb{E}_{\mathbb{P}}(\mathbf{d}) = \boldsymbol{\mu} & \\ \mathbb{E}_{\mathbb{P}}\left((d_{it} - \mu_{it})^2 \right) \le \sigma_{it}^2, & \forall i \in [N], t \in [T] \\ \mathbb{E}_{\mathbb{P}}\left(\left(\sum_{l=k}^{t} \mathbf{1}'(\mathbf{d}_l - \boldsymbol{\mu}_l) \right)^2 \right) \le \gamma_{kt}^2, & \forall k, t \in [T], k \le t \\ \mathbb{E}_{\mathbb{P}}\left((d_{ik} - \mu_{ik})(d_{jt} - \mu_{jt}) \right) \le \phi_{ikjt}^2, & \forall i, j \in [N], k, t \in [T] \\ \mathbb{P}\left(\mathbf{d} \in (\underline{\mathbf{d}}, \bar{\mathbf{d}}) \right) = 1 & \end{array} \right\}.$$

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