ECONOMIES AND GAMES WITH MANY AGENTS

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Declaration

I hereby declare that the thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

Sun Xiang

Sun, Xiang
May 12, 2013

To my parents,

my advisors,

and ...

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Contents

Summary

In this thesis, we consider three economic models with many agents, independent random partial matchings with general types, large games with actions in infinite-dimensional Banach spaces, and private information economies.

The deterministic cross-sectional type distribution in random matching models with a large population had been widely used in the economics literature. To obtain the deterministic type distribution, economists and geneticists have implicitly or explicitly assumed the independence condition and the law of large numbers for independent random matchings with a continuum population. However, the micro foundation for the formulation, the existence and the law of large numbers of independent random matchings with a continuum population had been lacking. Duffie and Sun (2007, 2012) firstly establish the micro foundation for the independent random universal matching and the independent ransom partial matching with finite types. In Chapter 3, we formally formulate the independent random partial matching with general types, establish its existence, and show the exact law of large numbers of it.

It is common sense that pure-strategy Nash equilibria may not exist in general noncooperative games. However, it is important from a game-theoretical point of view to know when pure-strategy Nash equilibria exist. For games with a nonatomic measure-theoretical structure and an uncountable compact metric action space, when the players' payoffs depend on their own actions and the action distribution of other players, there are several subtle possibilities; see Khan, Rath and Sun (1997), Khan, Rath and Sun (1999), Khan and Sun (1999), Keisler and Sun (2009) and Rath (1992) for details. The purpose of Chapter 4 is to consider the pure-strategy Nash equilibria for games with a nonatomic player space and an uncountable compact action set in an infinite-dimensional Banach space, where players' payoffs depend on their own actions and the average action of other players.

One important topic in general equilibrium analysis is the incentive compatibility of various solution concepts. In Chapter 5, we consider three solution concepts in a private information economy, *e.g.*, Radner equilibrium, private core and insurance equilibrium, and show that they are not incentive compatible.

Chapter 1

Introduction

Every economic model involves economic agents. When a model considers a fixed finite number of agents, the most natural agent space is the set $\{1, 2, \ldots, n\}$ for some positive integer n. In a vast literature in economics, one also needs to model the interaction of many agents in order to discover mass phenomena that do not necessarily occur in the case of a fixed finite number of agents. As pointed out by von Neumann and Morgenstern (1953),

> When the number of participants becomes really great, some hope emerges that the influence of every particular participant will become negligible, and that the above difficulties may recede and a more conventional theory become possible. Indeed, this was the starting point of much of what is best in economic theory.

For more discussion of mass phenomena in economics, see Khan and Sun (2002). A well-known example is the Edegeworth conjecture that the set of core allocations will shrink to the set of competitive equilibria as the number of agents goes to infinity though the former set is in general strictly bigger than the latter set for an economy with a fixed finite number of agents.[1]

To avoid complicated combinatorial arguments that may involve multiple steps of approximations for a large but finite number of agents, it is natural to consider economic models with an infinite number of agents. The mathematical abstraction of an atomless (countably-additive) measure space of agents provides a convenient idealization for a

[1] See Debreu and Scarf (1963).

large but finite number of agents. The archetype space in such a setting is the classical Lebesgue unit interval. That is why a general atomless measure space of agents is often referred to as a continuum of agents in a huge economics literature.

In this thesis, we will present three economics models, where the agent spaces are modeled by atomless probability spaces: independent random partial matchings with general types, large games with actions in infinite-dimensional Banach spaces, and private information economies.

1.1 Independent random partial matching

The deterministic cross-sectional type distribution in random matching models with a large population had been widely used in the economics literature. To obtain the deterministic type distribution, economists and geneticists have implicitly or explicitly assumed the independence condition and law of large numbers for independent random matchings with a continuum population. However, the micro foundation for the formulation, the existence and the law of large numbers of independent random matchings with a continuum population had been lacking.

To resolve the problem above, Duffie and Sun (2007, 2012) propose a condition of independence-in-types, and formulate independent random matchings for both static and dynamic cases and for both full matchings and partial matchings. In Duffie and Sun (2012), they prove the exact law of large numbers for independent random full matchings with general types and for independent random partial matchings with finite types in the static case, which follows immediately from the general exact law of large numbers in Sun (2006); see Theorems 1 and 2 in Duffie and Sun (2012) respectively.

The first theoretical treatment of the existence of independent random matchings with a continuum population is provided by Duffie and Sun (2007). In particular, in the static case, Duffie and Sun (2007) show the existence of independent universal (*i.e.*, type-free) random full matchings, and the existence of independent random partial matchings with finite types; see Theorems 2.4 and 2.6 therein respectively. Note that the proof of the latter existence result strictly depends on the finiteness of type space, so the formulation and the existence for independent random partial matchings with general types would require a more general setup.

In Duffie, Malamud and Manso (2012), the authors consider a discrete-time dynamic random matching model, where in each period it is exactly an independent random partial matching with the type space $\mathbb{R}$. The type for each agent characterizes the information she obtained, and will change after matching and trading. Upon matching, the two agents are given the opportunity to trade one unit of the asset in a double auction. Since there is no trading for agents with the same preferences, the authors assume that the matching probability for two agents with the same preference is zero, and the no-match probability is indeed the proportion of the agents with same preferences. By assuming the exact law of large numbers, Duffie, Malamud and Manso (2012) find the cross-sectional type distribution (density) after each period given the initial type distribution (density). In Molico (2006), another discrete-time dynamic random matching model is considered, where the population is represented by $[0, 1]$ and the type space is $[0, \infty)$. In this model, the type for each agent is given by her/his money holdings which is nonnegative. In every period agents are randomly and bilaterally matched, and an agent meets a potential trading partner with probability α, which will produce a random partial matching. By implicitly postulating the exact law of large numbers, Molico (2006) get the law of money motion.

The purpose of Chapter 3 is to provide a micro foundation for the formulation, the existence and the exact law of large numbers of independent random partial matchings with a continuum population and general types in the static case. In Theorem 3.2.3, we construct a joint agent-probability space that satisfies Duffie and Sun's independence-in-types condition. Though the existence result in Theorem 3.2.3 is stated using common measure-theoretic terms, its proof makes extensive use of nonstandard analysis. In particular, we construct a hyperfinite agent space, take the liftings for the initial type distribution and no-match probability function, transfer to the hyperfinite setting, and then work with a hyperfinite type space. Since the classical Lebesgue unit interval is an archetype agent space for economic models with a continuum of agents, we show that one can also take an extension of the classical Lebesgue unit interval as the agent space for independent random partial matchings in the static case; see Theorem 3.2.4, which is a generalization of Corollary 1 in Duffie and Sun (2012). Under Duffie and Sun's independence-in-types condition, the exact law of large numbers for independent random partial matchings with a continuum population and general types also follows immediately from the general exact law of large numbers in Sun (2006); see Proposition 3.3.1.

1.2 Nonatomic games with infinite-dimensional action spaces

It is common sense that pure-strategy Nash equilibria may not exist in general noncooperative games. However, it is important from a game-theoretical point of view to know when pure-strategy Nash equilibria exist. For a finite-player game, the existence of pure-strategy Nash equilibria follows from certain conditions on the payoff functions and strategy spaces. For games with a nonatomic measure-theoretic structure that models the space of players or information, a general purification principle due to Dvoretsky, Wald and Wolfowitz (1951a) guarantees that one can always obtain a pure-strategy Nash equilibrium from a mixed-strategy Nash equilibrium, when the action space is finite; see Dvoretsky, Wald and Wolfowitz (1951b), Khan, Rath and Sun (2006) and their references. For games with countable actions, similar results on pure-strategy Nash equilibria can be found in Khan and Sun (1995).

For games with a nonatomic measure-theoretical structure and an uncountable compact metric action space, when the players' payoffs depend on their own actions and the action distribution of other players, there are several subtle possibilities. First, when the space of players or information is modeled by the Lebesgue unit interval, counterexamples are constructed to show the nonexistence of pure-strategy Nash equilibria; see Khan, Rath and Sun (1997, 1999). Second, when the Lebesgue unit interval is replaced by a nonatomic Loeb space, positive results on pure-strategy Nash equilibria are shown in Khan and Sun (1999). Third, for a fixed nonatomic player space, it is shown in Keisler and Sun (2009) that any game with the given player space has a pure-strategy Nash equilibrium if and only if the underlying player space is saturated in the sense that any subspace is not countably generated modulo the null sets.

The purpose of Chapter 4 is to consider the pure-strategy Nash equilibria for games with a nonatomic player space and an uncountable compact action set in an infinite-dimensional Banach space, where players' payoffs depend on their own actions and the average action of other players. As shown in Khan, Rath and Sun (1997), when the player space is the Lebesgue unit interval and the action space is an uncountable compact subset of the Hilbert space ℓ_2—the space of square-summable real-valued sequences, pure-strategy Nash equilibria may not exist. Since various infinite-dimensional Banach spaces are widely used in the economics literature, a natural question is whether we could find a right infinite-dimensional Banach space rather than ℓ_2 to deliver a positive result on the existence of the pure-strategy Nash equilibria. We show that this is impossible as long as the player space is the Lebesgue unit interval. In particular, given any infinite-dimensional Banach space, there always exist nonatomic games with an uncountable compact action set in this Banach space such that these games do not have pure-strategy Nash equilibria, provided that the player space is the Lebesgue unit interval.

Nevertheless, if the player space is not the Lebesgue unit interval, it is possible to deliver a positive result on pure-strategy Nash equilibria for nonatomic games with infinite-dimensional action spaces. Khan and Sun (1999) show that when the Lebesgue unit interval is replaced by a nonatomic Loeb space, there exists a pure-strategy Nash equilibrium for any nonatomic game with any uncountable compact action set in an infinite-dimensional Banach space. It follows from the existence result in Khan and Sun (1999) and general saturation property that the existence result of pure-strategy Nash equilibria still holds when the player space is modeled by a saturated probability space.

A further and more interesting question is whether the converse of the above result is true. We provide an answer in the affirmative. In particular, we show that given a nonatomic player space and a fixed compact subset of a fixed infinite-dimensional Banach space, if every game with this compact subset as the common action set has a pure-strategy Nash equilibrium, then the underlying player space must be a saturated probability space; see Theorem 4.4.7. Put differently, if the player space is not a saturated probability space, then one can always construct a nonatomic game with this player space where players take actions from a given infinite-dimensional Banach space, such that it has no pure-strategy Nash equilibrium. It is worthwhile to note that our necessity result is not implied by the necessity part of Theorem 4.6 in Keisler and Sun (2009).

To summarize, to obtain a positive result on the existence of pure-strategy Nash equilibria for nonatomic games with actions in infinite-dimensional spaces, the measure-theoretic structure of the player space plays a fundamental role. It is worth noting that as far as the above counterexamples on Lebesgue interval are concerned, to guarantee the existence of pure-strategy Nash equilibria, one is not necessary to turn to saturated probability spaces, a simple extension of the Lebesgue unit interval does serve this purpose; see Proposition 4.5.1 below.

1.3 Private information economy

Economic decisions are essentially made based on a decision maker's vision of the future. The future is not known yet, hence all decisions are made with some degree of uncertainty. However, these decisions are not made entirely blindfolded. Agents rely on available information in plotting future plan, and information is asymmetric to agents. The classical Arrow-Debreu-McKenzie model has been extended to reflect these two facts, namely, uncertainty and informational asymmetry.

The first attempt to introduce uncertainty in Arrow-Debreu-McKenzie model was made by Arrow (1964) and Debreu (1959), who introduced a state-contingent claims model in which agents' utility function and initial endowment are contingent on the underlying state of nature. By treating a same commodity in two states of nature as different types of commodities, their model can be naturally mapped to a deterministic economy model to which standard techniques and results apply.

Radner (1968) further extended Arrow-Debreu's model to allow for asymmetric information. In Radner's model, each agent possesses a piece of private information which partially reveals the true state of nature. While Radner's model has the feature of uncertainty and informational asymmetry, no genuine perfect competition exists for each individual agent has non-negligible influence in such a finite-agent model.

Based on Radner's private information economy model and Aumann's large deterministic economy model (see Aumann (1964)), Sun and Yannelis (see Sun (2006), Sun and Yannelis (2007a), Sun and Yannelis (2008a)) introduced a private information economy model with a continuum of agents. In the model, agents have no direct knowledge of the underlying uncertainty. Instead, they are informed of a noisy private information signal giving them a clue about the real state of nature. Informational negligibility prevails in their model.

For the private information economy model, various solution concepts have been put forward that parallel the standard notions in a deterministic economy model. Radner (1968) introduced Radner equilibrium (a.k.a. Walrasian expectations equilibrium). In a Radner equilibrium, commodity prices vary over the states of nature. Each agent makes a state contingent consumption plan to maximize her expected utility, subject to her interim budget set. While Radner's notion of equilibrium has been unanimously accepted in the literature as the extension of the classic Walrasian equilibrium for the private information economy model, the situation is more complicated with the notion of core.

The complication is mainly due to the fact that in a private information economy, members of a coalition may exchange information for their good. Several definitions of core for the private information economy model have thus been proposed depending on the amount of information to be shared in a coalition. Wilson (1978) (see also Kobayashi (1980)) introduced the notion of coarse core with a minimal use of information that is common to all coalition members. Yannelis (1991) formulated the concept of private core in which each agent uses, and is limited to, her/his own private information.

Another notion of equilibrium that also deserves some attention is the so-called insurance equilibrium. This equilibrium is used to study insurance systems where each agent takes on individual risks and makes choices of consumption to spread risks across states of nature. In the insurance equilibrium model, agents can transfer income from one state to another through insurance against mishaps in the future. Therefore, in the model, an agent's budget set is not limited to the income in each state. This model was studied in the large finite-agent setting by Malinvaud (1972) and the continuum agent setting by Sun (2006). The latter paper further investigated the issue of insurability in a economy with a continuum of agents and obtained a characterization of insurable risks – individual risks are insurable if and only if they are essentially pairwise independent.

In the private information economy with finite agents, the solution concepts are not equivalent. However, it is well-known that in a deterministic economy model, although solution concepts are defined from different perspectives, they may coincide with each other under certain assumptions. For instance, Aumann (1964) showed the equivalence between Walrasian equilibrium and core in a large deterministic game. Sun *et al.* (2013) examines the above-mentioned concepts and shows that the same equivalence relationship continues to hold in the context of private information economy with a continuum of agents. Note that in the private information economy with a continuum of agents, besides the above-mentioned solution concepts, there are some others, *e.g.*, ex ante efficient core and ex post efficient core, and the equivalence may not still hold.

In the private information economy with finite agents, the solution concepts above-mentioned are automatically incentive compatible, and in the private information economy with a continuum of agents, the ex ante efficient core allocation is also incentive compatible; see Sun and Yannelis (2008a). In this chapter, we will see that the private core allocation is not always incentive compatible (so are Radner equilibrium and insurance equilibrium). Compared with the private information economy with finite agents, this issue comes from the resource feasibility. In the private information economy with finite agents, feasibility is a restriction for the available allocation. However, in the private information economy with a continuum of agents, feasibility immediately follows the law of large numbers, and no allocation will be precluded by feasibility.

1.4 Organization

The main results in Chapters 3, 4 and 5 are based on the papers Sun (2013a), Sun and Zhang (2013) and Sun *et al.* (2013) respectively.

This thesis is organized as follows. In Chapter 2, we present the general exact law of large numbers, the saturated probability spaces and related definitions and properties. Chapter 3 provide a micro foundation for independent random partial matchings with a continuum population and general types. In Chapter 4, we study the existence of pure-strategy Nash equilibria for nonatomic games where players take actions in an infinite-dimensional Banach space. In Chapter 5, we consider the incentive compatibility of three equivalent solution concepts, Radner equilibrium, private core and insurance equilibrium, in a private information economy. Some concluding remarks are discussed in Chapter 6.

Chapter 2

Mathematical Preliminaries

A Polish space means a complete separable metric space. For a Polish space X, denote its Borel σ-algebra by $\mathcal{B}_X$, and by $\mathcal{M}(X)$ the space of all Borel probability measures on X with the Prohorov metric ρ. We recall that $\mathcal{M}(X)$ is again a Polish space, $\mathcal{M}(X)$ has the topology of weak convergence, and if X is compact then so is $\mathcal{M}(X)$.

Throughout this thesis, we will use the convention that a probability space is always a complete, countably additive measure space. A probability space $(I, \mathcal{I}, \lambda)$ is atomless (or nonatomic) if there does not exist $A \in \mathcal{I}$ such that $\lambda(A) > 0$, and for any $\mathcal{I}$-measurable subset C of A, $\lambda(C) = 0$ or $\lambda(C) = \lambda(A)$. Given a subset $\mathcal{C}$ of $\mathcal{I}$, denote by $\sigma(\mathcal{C})$ the smallest σ-algebra containing $\mathcal{C}$.

For any subset $A \in \mathcal{I}$ with $\lambda(A) > 0$, denote by $(A, \mathcal{I}^A, \lambda^A)$ the probability space restricted to A, where $\mathcal{I}^A$ is the σ-algebra $\{C \in \mathcal{I} \colon C \subseteq A\}$, and λ^A is the probability measure rescaled from the restriction of λ to $\mathcal{I}^A$. Moreover, for any $\mathcal{I}$-measurable function f from I to a Polish space X, f^A is the restriction of f to A.

In this chapter we will present the exact law of large numbers and related concepts in Section 2.1, and then concepts of saturated probability spaces in Section 2.2.

2.1 Fubini extension, essentially pairwise independence, and the exact law of large numbers

Let probability spaces $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, \mathbf{P})$ be our index and sample spaces, respectively. Let $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes \mathbf{P})$ be the usual product probability space. For a function f on $I \times \Omega$ (not necessarily $\mathcal{I} \otimes \mathcal{F}$-measurable), and for $(i, \omega) \in I \times \Omega$, f_i represents the function $f(i, \cdot)$ on Ω, and f_ω the function $f(\cdot, \omega)$ on I.

In order to work with independent processes arising from economies and games with infinitely many agents, we need to work with an extension of the usual measure-theoretic product that retains the Fubini property. A formal definition, as in Sun (2006), is as follows.

Definition 2.1.1. *A probability space $(I \times \Omega, \mathcal{W}, \mathbf{Q})$ extending the usual product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes \mathbf{P})$ is said to ba a* ***Fubini extension*** *of $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes \mathbf{P})$ if for any real-valued $\mathbf{Q}$-integrable function f on $(I \times \Omega, \mathcal{W})$,*

1. *The two functions f_i and f_ω are integrable, respectively, on $(\Omega, \mathcal{F}, \mathbf{P})$ for λ-almost all $i \in I$, and on $(I, \mathcal{I}, \lambda)$ for $\mathbf{P}$-almost all $\omega \in \Omega$;*

2. *$\int_\Omega f_i \, \mathrm{d}\mathbf{P}$ and $\int_I f_\omega \, \mathrm{d}\lambda$ are integrable, respectively, on $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, \mathbf{P})$, with $\int_{I \times \Omega} f \, \mathrm{d}\mathbf{Q} = \int_I (\int_\Omega f_i \, \mathrm{d}\mathbf{P}) \, \mathrm{d}\lambda = \int_\Omega (\int_I f_\omega \, \mathrm{d}\lambda) \, \mathrm{d}\mathbf{P}$.*

To reflect the fact that the probability space $(I \times \Omega, \mathcal{W}, \mathbf{Q})$ has $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, \mathbf{P})$ as its marginal spaces, as required by the Fubini property, it will be denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$.

We now introduce the following crucial independence condition, defined by Sun (2006). We state the definition using a Polish space X for the sake of generality.

Definition 2.1.2. *An $\mathcal{I} \boxtimes \mathcal{F}$-measurable process f from $I \times \Omega$ to a Polish space X is said to be* ***essentially pairwise independent*** *if for λ-almost all $i \in I$, the random variables f_i and f_j are independent for λ-almost all $j \in I$.*

Sun (2006) establishes the following theorem, the exact law of large numbers (in sample distribution) and its converse:

Fact 2.1.3 (Theorem 2.8 in Sun (2006))**.** *Let f be a process from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ to a Polish space X. Then the following are equivalent.*

1. *The random variables f_i are essentially pairwise independent.*

2. *For any set $A \in \mathcal{I}$ with $\lambda(A) > 0$, the sample distribution $\lambda(f_\omega^A)^{-1}$ is the same as the distribution $(\lambda^A \boxtimes \mathbf{P})(f^A)^{-1}$ of the process f^A for $\mathbf{P}$-almost all $\omega \in \Omega$, where f^A is the restriction of f to $A \times \Omega$, $\mathcal{I}^A = \{C \in \mathcal{I} : C \subseteq A\}$ and $(\mathcal{I}^A \boxtimes \mathcal{F}) = \{E \in \mathcal{I} \boxtimes \mathcal{F} : E \subset (A \times \Omega)\}$, and λ^A and $(\lambda^A \boxtimes \mathbf{P})$ the probability measures rescaled respectively from the restrictions of λ to $\mathcal{I}^A$ and $(\lambda \boxtimes \mathbf{P})$ to $(\mathcal{I}^A \boxtimes \mathcal{F})$.*

2.2 Saturated probability space

The definition of saturated probability spaces introduced by Hoover and Keisler (1984) is presented as follows:

Definition 2.2.1. *1. A probability space $(T, \mathcal{T}, \mu)$ is said to satisfy the **saturation property** for a measure $\tau \in \mathcal{M}(X \times Y)$ if for every $\mathcal{T}$-measurable function $f : I \to X$ with the distribution $\mu \circ f^{-1} = \mathrm{marg}_X(\tau)$, there exists another $\mathcal{T}$-measurable function $g : T \to Y$ such that $\mu \circ (f, g)^{-1} = \tau$, where $\mathrm{marg}_X(\tau)$ is the marginal of τ in $\mathcal{M}(X)$.*

*2. A probability space $(T, \mathcal{T}, \mu)$ is **saturated** (or has full saturation) if $(T, \mathcal{T}, \mu)$ is atomless, and for every pair of Polish spaces X and Y, $(T, \mathcal{T}, \mu)$ satisfies the saturation property for every $\tau \in \mathcal{M}(X \times Y)$.*

The saturated probability space has many equivalent characterizations, *e.g.*, the $\aleph_1$-atomless probability space (see Hoover and Keisler (1984)), the nowhere countably-generated probability space (see Loeb and Sun (2009)), and the probability space whose Maharam spectrum is a set of uncountable cardinals (see Fajardo and Keisler (2002)).

The following concept, $\aleph_1$-atomless probability spaces, is proposed by Hoover and Keisler (1984).

Definition 2.2.2. *Let $(T, \mathcal{T}, \mu)$ be a probability space.*

1. *Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{T}$, we say that $\mathcal{T}$ is **atomless over** $\mathcal{A}$, if for every $D \in \mathcal{T}$ with $\mu(D) > 0$ there is a $\mathcal{T}$-measurable subset $D_0 \subseteq D$, such that on some set of positive probability,*

$$0 < P[D_0|\mathcal{A}] < P[D|\mathcal{A}],$$

where $P[D|\mathcal{A}]$ is the conditional probability of D with respect to the σ-algebra $\mathcal{A}$. A σ-algebra is ***atomless*** *if it is atomless over the trivial a-algebra.*

2. *We say $\mathcal{T}$ is* ***$\aleph_1$-atomless*** *if $\mathcal{T}$ is atomless over every $\mathcal{A}$ which is countably generated.*

Definition 2.2.3. *A probability space $(T, \mathcal{T}, \mu)$ is called* ***countably generated (modulo the null sets)*** *(or essentially countably generated) if there is a countable set $\{A_n \in \mathcal{T} : n \in \mathbb{N}\}$ such that for any $S \in \mathcal{T}$, there is a set S' in the σ-algebra generated by $\{A_n \in \mathcal{T} : n \in \mathbb{N}\}$ with $\mu(S \triangle S') = 0$, where $\triangle$ denotes the symmetric difference in $\mathcal{T}$. A probability space $(T, \mathcal{T}, \mu)$ is said to be* ***nowhere countably generated*** *if for any subset $S \in \mathcal{T}$ with $\mu(S) > 0$, the rescaled probability space $(S, \mathcal{T}^S, \mu^S)$ is not countably generated.*

Before introducing the probability space whose Maharam spectrum is a set of uncountable cardinals, we need some preparation on measure algebra.

Let $(T, \mathcal{T}, \mu)$ be a probability space. Consider a relation '$\sim$' on $\mathcal{T}$ as follows, for any $E, F \in \mathcal{T}$, $E \sim F$ if and only if $\mu(E \triangle F) = 0$, where $\triangle$ denotes the symmetric difference. It is clear that $\sim$ is an equivalence relation on $\mathcal{T}$. For any $E \in \mathcal{T}$, let $\hat{E} = \{F \in \mathcal{T} : F \sim E\}$ be the equivalence class of E, and clearly $E \in \hat{E}$. Th pair $(\hat{\mathcal{T}}, \hat{\mu})$ is said to be the **measure algebra** of $(T, \mathcal{T}, \mu)$, where $\hat{\mathcal{T}}$ is the quotient Boolean algebra for the equivalence relation $\sim$, *i.e.*, the set of equivalence classes in $\mathcal{T}$ for $\sim$, and $\hat{\mu} : \hat{\mathcal{T}} \to [0, 1]$ is given by $\hat{\mu}(\hat{E}) = \mu(E)$, for some $E \in \hat{E}$.

If μ_1 and μ_2 are probability measures on disjoint sample spaces T_1 and T_2 respectively, $1 > \alpha > 0$, then the convex combination $\alpha \cdot \mu_1 + (1 - \alpha) \cdot \mu_2$ is the probability space on $T_1 \cup T_2$ formed in the obvious way with μ_1 and μ_2 having probabilities α and $1 - \alpha$. Convex combinations of measure algebras, and countable convex combinations of probability spaces and of measure algebras, are defined in an analogous manner. Let $[0, 1]^\kappa$ be the probability space formed by taking the product measure of κ copies of the space $[0, 1]$ with the Lebesgue measure. The measure algebras of the spaces $[0, 1]^\kappa$ are of special importance, and are called **homogeneous measure algebras**. The fundamental theorem about measure algebras in Maharam (1942) shows that there are very few measure algebras.

Fact 2.2.4 (Theorem of Maharam (1942))**.** *For every atomless probability space $(T, \mathcal{T}, \mu)$, there is a finite or countable set of distinct cardinals $\{\kappa_i\}$ such that the measure algebra of $(T, \mathcal{T}, \mu)$ is a convex combination of the homogeneous measure algebras $[0, 1]^{\kappa_i}$.*

The set of cardinals $\{\kappa_i\}$ in Maharam's Theorem is clearly unique. This set is called the **Maharam spectrum** of $(T, \mathcal{T}, \mu)$.

Fact 2.2.5. *For each atomless probability space $(T, \mathcal{T}, \mu)$, the following are equivalent:*

1. *$(T, \mathcal{T}, \mu)$ is saturated.*
2. *$(T, \mathcal{T}, \mu)$ is $\aleph_1$-atomless.*
3. *$(T, \mathcal{T}, \mu)$ is nowhere countably generated.*
4. *The Maharam spectrum of $(T, \mathcal{T}, \mu)$ is a set of uncountable cardinals.*

Proof. The equivalence of (1) and (2) is proved in Corollary 4.5(i) of Hoover and Keisler (1984). The equivalence of (3) and (4) follows from Maharam's Theorem (see Fact 2.2.4). A direct proof that (1) is equivalent to (4) is also given in Theorem 3B.7 of Fajardo and Keisler (2002). □

Remark 2.2.6. *It is well-known that the Lebesgue unit interval, denoted by $(L, \mathcal{L}, \eta)$, is countably generated modulo the null sets, and hence not saturated. In contrast, any atomless Loeb probability space is saturated; see Hoover and Keisler (1984). One can also extend the Lebesgue unit interval into a saturated probability space, see Kakutani (1944), Section 6 in Podczeck (2008) and Sun and Zhang (2009); furthermore, it is worth to note that, the construction of a saturated extension of the Lebesgue unit interval in Sun and Zhang (2009) is not an issue, while the key is to construct a rich Fubini extension based on this extended Lebesgue interval.*

Remark 2.2.7. *The class of saturated probability spaces is first formally introduced by Hoover and Keisler (1984), and developed by Fajardo and Keisler (2002) and Keisler and Sun (2002, 2009). Besides "$\aleph_1$-atomless spaces", "nowhere countably-generated spaces" and "spaces whose Maharam spectrum is a set of uncountable cardinals", they also have other names in literatures, e.g., "nowhere separable spaces" in Džamonja and Kunen (1995), "rich probability spaces" in Keisler (1997), and Noguchi (2009), and "super-atomless spaces" in Podczeck (2008).*

Fact 2.2.8. *If $(T, \mathcal{T}, \mu)$ is a saturated probability space, then any other probability space whose measure algebra is isomorphic to $(\hat{\mathcal{T}}, \hat{\mu})$ is also saturated.*

Chapter 3

Independent random partial matching with general types

3.1 Introduction

The deterministic cross-sectional type distribution in random matching models with a large population had been widely used in the economics literature. Some models consider random matchings with finite types, *e.g.*, Hardy (1908), Kiyotaki and Wright (1993) and Duffie, Gârleanu and Pedersen (2005). On the other hand, for a wide class of random matching models with a large population, it is impossible to capture the relevant properties within a finite type space; for example, Green and Zhou (2002), Molico (2006) and Duffie, Malamud and Manso (2012) choose the intervals $[0, 1]$, $[0, \infty)$ and the real line $\mathbb{R}$ as the type space, respectively. Duffie and Sun (2012) also discuss extensive references within general equilibrium theory, game theory, monetary theory, labor economics, illiquid financial markets and biology. Additional references for matching with general types include Shi (1997), Lagos and Wright (2005), Zhu (2003, 2005) and Mailath *et al.* (2012).

To obtain the deterministic type distribution, economists and geneticists have implicitly or explicitly assumed the independence condition[1] and law of large numbers for independent random matchings with a continuum population. Hardy (1908) is the first, to our knowledge, to study the random matchings with a large population. In his paper,

[1]Roughly speaking, by independence condition, we mean that for distinct persons i and j, i's matching is independent of j's matching. The precise definition will be given in Definition 3.2.1.

Hardy (1908) proposed that with random matching in a large population, one could determine the constant fractions of each type in the population. In fact, Hardy (1908) implicitly assumed the independence condition, and then applied informally a law of large numbers for random matchings to deduce his results. However, the micro foundation for the formulation, the existence and the law of large numbers of independent random matchings with a continuum population had been lacking.

To resolve the problem above, Duffie and Sun (2007, 2012) propose a condition of independence-in-types, and formulate independent random matchings for both static and dynamic cases and for both full matchings and partial matchings. In Duffie and Sun (2012), they prove the exact law of large numbers for independent random full matchings with general types and for independent random partial matchings with finite types in the static case,[2] which follows immediately from the general exact law of large numbers in Sun (2006); see Theorems 1 and 2 in Duffie and Sun (2012) respectively. Note that the independence condition is a general behavioural assumption. When agents choose their partners without coordinations among themselves, it is reasonable to assume independence for the underlying random matching. Furthermore, it should be necessary to distinguish an *ad hoc* example with some particular correlation structure on the random matching from a general result in the setting of the law of large numbers, where the deterministic type distribution in the random matching follows from the independence condition on the random matching. See Section 6 in Duffie and Sun (2012) for more detailed discussions on the *ad hoc* random matchings without independence.

The first theoretical treatment of the existence of independent random matchings with a continuum population is provided by Duffie and Sun (2007).[3] In particular, in the static case, Duffie and Sun (2007) show the existence of independent universal (*i.e.*, type-free) random full matchings,[4] and the existence of independent random partial matchings with finite types; see Theorems 2.4 and 2.6 therein respectively. Note that the proof of the latter existence result strictly depends on the finiteness of type space, so the formulation and the existence for independent random partial matchings with

[2] Duffie and Sun (2012) also prove the exact law of large numbers for independent random matchings in the dynamic case, which is beyond the scope of this chapter; see Theorem 3 in Duffie and Sun (2012).

[3] One should note that the exact law of large numbers and its corollary deterministic cross-sectional type distribution for independent random matchings will make no sense if such models do not exist.

[4] The random matching is universal in the sense that it does not depend on particular type functions. Moreover, this result implies the existence of an independent universal random full matching model that satisfies a few strong conditions that are specified in Footnote 4 of McLennan and Sonnenschein (1991). Podczeck and Puzzello (2012) give an alternative proof for the existence of independent universal random full matchings with a continuum population in the static case.

general types would require a more general setup.

In Duffie, Malamud and Manso (2012), the authors consider a discrete-time dynamic random matching model, where in each period it is exactly an independent random partial matching with the type space $\mathbb{R}$. The type for each agent characterizes the information she obtained, and will change after matching and trading. Upon matching, the two agents are given the opportunity to trade one unit of the asset in a double auction. Since there is no trading for agents with the same preferences, the authors assume that the matching probability for two agents with the same preference is zero, and the no-match probability is indeed the proportion of the agents with same preferences. By assuming the exact law of large numbers, Duffie, Malamud and Manso (2012) find the cross-sectional type distribution (density) after each period given the initial type distribution (density). In Molico (2006), another discrete-time dynamic random matching model is considered, where the population is represented by $[0, 1]$ and the type space is $[0, \infty)$. In this model, the type for each agent is given by her/his money holdings which is nonnegative. In every period agents are randomly and bilaterally matched, and an agent meets a potential trading partner with probability α, which will produce a random partial matching. By implicitly postulating the exact law of large numbers, Molico (2006) get the law of money motion.

The purpose of this chapter is to provide a micro foundation for the formulation, the existence and the exact law of large numbers of independent random partial matchings with a continuum population and general types in the static case. In Theorem 3.2.3, we construct a joint agent-probability space that satisfies Duffie and Sun's independence-in-types condition. Though the existence result in Theorem 3.2.3 is stated using common measure-theoretic terms, its proof makes extensive use of nonstandard analysis. In particular, we construct a hyperfinite agent space, take the liftings for the initial type distribution and no-match probability function, transfer to the hyperfinite setting, and then work with a hyperfinite type space.[5] Since the classical Lebesgue unit interval is an archetype agent space for economic models with a continuum of agents, we show that one can also take an extension of the classical Lebesgue unit interval as the agent space for independent random partial matchings in the static case; see Theorem 3.2.4,

[5]It is a well-known property that hyperfinite probability spaces capture the asymptotic properties of large but finite probability spaces, so the use of such a probability space does provide some advantages. Brown and Robinson (1975) introduce the application of nonstandard analysis into economics. For recent applications of nonstandard analysis in economics, see also Anderson and Raimondo (2008), Khan and Sun (2002) and Sun and Yannelis (2007b, 2008a). One can pick up some background knowledge on nonstandard analysis from the first three chapters of the book Loeb and Wolff (2000).

which is a generalization of Corollary 1 in Duffie and Sun (2012). Under Duffie and Sun's independence-in-types condition, the exact law of large numbers for independent random partial matchings with a continuum population and general types also follows immediately from the general exact law of large numbers in Sun (2006); see Proposition 3.3.1.

The remainder of the chapter is organized as follows. Section 3.2 provides the definition of independent random partial matchings with a continuum population and general types, and discusses its existence in Theorem 3.2.3. Theorem 3.2.4 proves the existence of independent random partial matchings with general types, where the agent space is an extension of the Lebesgue unit interval. In Section 3.3, Proposition 3.3.1 shows the exact law of large numbers for independent random partial matchings. Proofs of the main results will be given in Section 3.4.

3.2 The existence of independent random partial matchings

Let probability spaces $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, \mathbf{P})$ be our index and sample spaces, respectively. In our applications, $(I, \mathcal{I}, \lambda)$ is an atomless probability space that indexes the agents. Let $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes \mathbf{P})$ and $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ be the usual product probability space and Fubini extension respectively. Below is the formal definition of independent random partial matchings with a continuum population and general types in the static case.

Definition 3.2.1 (Independent random partial matchings with general types). *Let a Polish space S be the set of types, $\mathcal{S}$ the σ-algebra of all Borel measurable subsets of S.*

Let $\alpha\colon I \to S$ be an $\mathcal{I}$-measurable type function with type distribution p on S, that is, for every $B \in \mathcal{S}$, $p(B) = \lambda(\alpha^{-1}(B))$. Let $q\colon S \to [0,1]$ be an $\mathcal{S}$-measurable no-match probability function, that is, for every $k \in S$, $q(k)$ is the no-match probability for an agent whose type is k.

Given a subset $I' \subset I$, a full matching ϕ on I' is a bijection from I' to I' such that for each $i \in I'$, $\phi(i) \neq i$ and $\phi^2(i) = i$.

Let π be a mapping from $I \times \Omega$ to $I \cup \{J\}$, where J denotes "no-match".

1. *We say that π is a random partial matching with $\mathcal{S}$-measurable no-match probability*

function q if:

(a) *For each $\omega \in \Omega$, the restriction of π_ω to $I - \pi_\omega^{-1}(\{J\})$ is a full matching on $I - \pi_\omega^{-1}(\{J\})$;*

(b) *After extending the type function α to $I \cup \{J\}$ so that $\alpha(J) = J$, and letting g be the type process $\alpha(\pi)$, we have g measurable from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ to $S \cup \{J\}$;*

(c) *For λ-almost all $i \in I$, $\mathbf{P}(g_i = J) = q(\alpha(i))$ and*

$$\mathbf{P}(g_i \in C) = [1 - q(\alpha(i))] \frac{\int_C [1 - q(k)] \, \mathrm{d}p(k)}{\int_S [1 - q(k)] \, \mathrm{d}p(k)}. \tag{3.1}$$

for any $C \in \mathcal{S}$.

2. *A random partial matching π is said to be independent in types if the type process g (taking values in $S \cup \{J\}$) is essentially pairwise independent.*

Condition 1-(a) of this definition says that an agent i with $\pi_\omega(i) = J$ is not matched, while any agent in $I - \pi_\omega^{-1}(\{J\})$ is matched. This produces a partial matching on I. Condition 1-(b) is the measurability requirement.

Condition 1-(c) means that if an agent i is matched, its probability of being matched to an agent, whose type is in the given type subset, should be proportional to the type distribution of matched agents. The fraction of the population of matched agents among the total population is $\int_S [1 - q(k)] \, \mathrm{d}p(k)$. Thus, the relative fraction of types-C matched agents[6] to that of all the matched agents is $\int_C [1-q(k)] \, \mathrm{d}p(k) / \int_S [1-q(k)] \, \mathrm{d}p(k)$. This implies that the probability that an agent i is matched to a types-C agent is $[1 - q(\alpha(i))] \int_C [1 - q(k)] \, \mathrm{d}p(k) / \int_S [1 - q(k)] \, \mathrm{d}p(k)$. When $\int_S [1 - q(k)] \, \mathrm{d}p(k) = 0$, almost no agents will be matched.

Condition 2 (*i.e.*, independence-in-types condition) says that for almost all agents $i, j \in I$, whether agent i is unmatched or matched to a types-C agent is independent of a similar event for agent j. Furthermore, this condition is weaker than pairwise/mutual independence since each agent is allowed to have correlation with a null set of agents (including finitely many agents since a finite set is null under an atomless measure).

Note that Condition 2 allows the application of the general exact law of large numbers in Sun (2006) to claim that the fraction of the total population consisting of types-

[6] By a types-C agent, we mean an agent whose type belongs to C.

B agents that are matched to types-C agents is aggregately proportional to the type distribution of matched agents. This result is formally stated in Proposition 3.3.1, whose proof is given in Section 3.4.4.

As indicated in the second paragraph on Page 1136 of Duffie and Sun (2012), the universal matching as constructed in the proof of Theorem 2.4 in Duffie and Sun (2007) also has the property that the mappings of the random partners of agents are not only pairwise independent as shown explicitly on Page 399 of Duffie and Sun (2007), but also mutually independent for finitely many different agents. Here we state the result formally and give its proof in Section 3.4.1.

Proposition 3.2.2. *There exists an atomless probability space $(I, \mathcal{I}, \lambda)$ of agents, a sample probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ of the usual product probability space, and a random full matching π from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ to I such that*

1. *(i) for each $\omega \in \Omega$, $\lambda(\pi_\omega^{-1}(A)) = \lambda(A)$ for any $A \in \mathcal{I}$, (ii) for each $i \in I$, $\mathbf{P}(\pi_i^{-1}(A)) = \lambda(A)$ for any $A \in \mathcal{I}$, (iii) for any $A_1, A_2 \in \mathcal{I}$, $\lambda(A_1 \cap \pi_\omega^{-1}(A_2)) = \lambda(A_1)\lambda(A_2)$ holds for $\mathbf{P}$-almost all $\omega \in \Omega$;*

2. *π is mutally independent, which means that for any distinct $i_1, i_2, \ldots, i_r \in I$, $(\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_r})\colon (\Omega, \mathcal{F}, \mathbf{P}) \to (\times^r I, \boxtimes^r \mathcal{I}, \boxtimes^r \lambda)$ is a measure-preserving mapping.*

Since the universal independent random matching as constructed in the proof of Theorem 2.4 in Duffie and Sun (2007) can be applied to any type functions (taking values in any finite or infinite space), there is no issue for independent random full matching with general types.[7] However, the formulations and the proofs for independent random partial matching in Duffie and Sun (2007, 2012) do rely on the finite types used there. Thus independent random partial matching for general types need to be treated separately. The following theorem generalizes Theorem 2.6 in Duffie and Sun (2007) to the case involving general types, whose proof is given in Section 3.4.2.[8]

[7]For any $\alpha\colon I \to S$, and any $B_1, B_2 \subset S$, item (iii) in Part 1 implies that $\lambda(\alpha^{-1}(B_1) \cap \pi_\omega^{-1}(\alpha^{-1}(B_2))) = \lambda(\alpha^{-1}(B_1))\lambda(\alpha^{-1}(B_2))$, which is useful for applications.

[8]The author thanks Darrell Duffie for the following remark. When the no-match probability function is constant, then the existence of independent random partial matching with general types follows immediately from the existence of independent universal random full matching by introducing an additional type to be interpreted as "no-match".

Theorem 3.2.3. *There is an atomless probability space* $(I, \mathcal{I}, \lambda)$ *of agents, such that for any given* $\mathcal{I}$*-measurable type function* α *from* I *to* S*, and for any given* $\mathcal{S}$*-measurable no-match probability function* q *from* S *to* $[0, 1]$*,*

1. *there exists a sample space* $(\Omega, \mathcal{F}, \mathbf{P})$*, and a Fubini extension* $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$*;*

2. *there exists an independent-in-types random partial matching* π *from* $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ *to* I *with* q *as the no-match probability function.*

In the following theorem, we will show the existence of the independent random partial matching with a continuum population and general types, where the agent space $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$ is an extension of the Lebesgue unit interval $(L, \mathcal{L}, \eta)$ in the sense that $\hat{I} = L = [0, 1]$, the σ-algebra $\hat{\mathcal{I}}$ contains the Lebesgue σ-algebra $\mathcal{L}$, and the restriction of $\hat{\lambda}$ to $\mathcal{L}$ is the Lebesgue measure η. Its proof is given in Section 3.4.3.

Theorem 3.2.4. *For any given type distribution* p *on* S*, and any given* $\mathcal{S}$*-measurable no-match probability function* q *from* S *to* $[0, 1]$*, there exists a Fubini extension* $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes \mathbf{P})$ *such that*

1. *the agent space* $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$ *is an extension of the Lebesgue unit interval* $(L, \mathcal{L}, \eta)$*.*

2. *there exists an independent-in-types random partial matching* π *from* $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes \mathbf{P})$ *to* $\hat{I}$ *with type distribution* p *and with* q *as the no-match probability function.*

3.3 The exact law of large numbers of independent random partial matchings

In the following, we will show the exact law of large numbers for independent random partial matchings with a continuum population and general types in the static case.

Proposition 3.3.1. *If* π *is an independent-in-types random partial matching from* $I \times \Omega$ *to* $I \cup \{J\}$ *with* $\mathcal{S}$*-measurable no-match probability function* q *from* S *to* $[0, 1]$*, then, for* $\mathbf{P}$*-almost all* $\omega \in \Omega$*:*

1. *For any type subset $B \in \mathcal{S}$, the fraction of the total population consisting of types-B agents that are unmatched is*

$$\lambda(\{i \in I \mid \alpha(i) \in B, g_\omega(i) = J\}) = \int_B q(k)\, \mathrm{d}p(k). \tag{3.2}$$

2. *For any type subsets $B, C \in \mathcal{S}$, the fraction of the total population consisting of types-B agents that are matched to types-C agents is*

$$\lambda(\{i \in I \mid \alpha(i) \in B, g_\omega(i) \in C\}) = \int_B [1 - q(k)]\, \mathrm{d}p(k) \frac{\int_C [1 - q(l)]\, \mathrm{d}p(l)}{\int_S [1 - q(l)]\, \mathrm{d}p(l)}. \tag{3.3}$$

3.4 Proofs

3.4.1 Proof of Proposition 3.2.2

Proof of Proposition 3.2.2. Fix an even hyperfinite natural number N in the set ${}^*\mathbb{N}_\infty$ of unlimited hyperfinite natural numbers. Let $I = \{1, 2, \ldots, N\}$, let $\mathcal{I}_0$ be the collection of all the internal subsets of I, and let λ_0 be the internal counting probability measure on $\mathcal{I}_0$. Let $(I, \mathcal{I}, \lambda)$ be the Loeb space of the internal probability space $(I, \mathcal{I}_0, \lambda_0)$. Note that $(I, \mathcal{I}, \lambda)$ is obviously atomless.

We can draw agents from I in pairs without replacement; and then match them in these pairs. The procedure can be the following. Take one fixed agent; this agent can be matched with $N-1$ different agents. After the first pair is matched, there are $N-2$ agents. We can do the same thing to match a second pair with $N-3$ possibilities. Continue this procedure to produce a total number of $1 \times 3 \times \cdots \times (N-3) \times (N-1)$, denoted by $(N-1)!!$, different matchings. Let Ω be the space of all such matchings, $\mathcal{F}_0$ the collection of all internal subsets of Ω, and $\mathbf{P}_0$ the internal counting probability measure on $\mathcal{F}_0$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the Loeb space of the internal probability space $(\Omega, \mathcal{F}_0, \mathbf{P}_0)$.

Let $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \lambda_0 \otimes \mathbf{P}_0)$ be the internal product probability space of $(I, \mathcal{I}_0, \lambda_0)$ and $(\Omega, \mathcal{F}_0, \mathbf{P}_0)$. Then $\mathcal{I}_0 \otimes \mathcal{F}_0$ is actually the collection of all the internal subsets of $I \times \Omega$ and $\lambda_0 \otimes \mathbf{P}_0$ is the internal counting probability measure on $\mathcal{I}_0 \otimes \mathcal{F}_0$. Let $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ be the Loeb space of the internal product $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \lambda_0 \otimes \mathbf{P}_0)$, which is indeed a Fubini extension of the usual product probability space.

Part (1) has already in Duffie and Sun (2007), and in the following we will focus on

Part (2).

For distinct $i_1, i_2, \ldots, i_r \in I$, consider the joint event

$$E = \big\{\omega \in \Omega : (\pi_{i_1}(\omega), \pi_{i_2}(\omega), \ldots, \pi_{i_r}(\omega)) = (j_1, j_2, \ldots, j_r)\big\},$$

that is, agent i_z is matched to agent j_z, $z = 1, 2, \ldots, r$. In order to show the measure-preserving property of the mapping $(\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_r})$ in the following paragraph, we need to know the value of $\mathbf{P}_0(E)$ in three different cases.

The first case is that there are $x \neq y \in \{1, 2, \ldots, r\}$, such that $j_x = j_y$. In this case, $\mathbf{P}_0(E) = 0$. Let D_1 denote the set $\{(j_1, j_2, \ldots, j_r)\colon j_x = j_y \text{ for some } x \neq y\}$

The second case is that there are $x, y \in \{1, 2, \ldots, r\}$, such that $i_x = j_y$. In this case, $\mathbf{P}_0(E) = \frac{1}{|N-1|}$. Let D_2 denote the set $\{(j_1, j_2, \ldots, j_r)\colon i_x = j_y \text{ for some } x, y\}$.

The third case applies if the indices $i_1, i_2, \ldots, i_r$ and $j_1, j_2, \ldots, j_r$ are completely distinct. In this third case, after the pairs $(i_1, j_1), (i_2, j_2), \ldots, (i_r, j_r)$ are drawn, there are $N-2r$ agents left, and hence there are $(N-2r-1)!!$ ways to draw the rest of the pairs in order to complete the matching. This means that $\mathbf{P}_0(E) = (N-2r-1)!!/(N-1)!! = 1/((N-1)(N-3)\cdots(N-2r+1))$.

Let $(\times^r I, \otimes^r \mathcal{I}_0, \otimes^r \lambda_0)$ be the internal product of r copies of $(I, \mathcal{I}_0, \lambda_0)$, and $(\times^r I, \boxtimes^r \mathcal{I}, \boxtimes^r \lambda)$ the Loeb space of the internal product. Fix any distinct $i_1, i_2, \ldots, i_r \in I$. The third case of the above paragraph implies that for any internal set $G \in \otimes^r \mathcal{I}_0$,

$$\mathbf{P}_0\big(\{\omega \in \Omega\colon (\pi_{i_1}(\omega), \ldots, \pi_{i_r}(\omega)) \in G - D_1 - D_2\}\big) \simeq \frac{|G|}{N^r} = \otimes^r \lambda_0(G) \tag{3.4}$$

By using the formula for $\mathbf{P}_0(E)$ in the first two cases, we can obtain that

$$\mathbf{P}_0\big(\{\omega \in \Omega\colon (\pi_{i_1}(\omega), \ldots, \pi_{i_r}(\omega)) \in D_1 \cup D_2\}\big) = \frac{1}{N-1} \simeq 0 \tag{3.5}$$

Equations (3.4) and (3.5) imply that

$$\mathbf{P}_0\big(\{\omega \in \Omega\colon (\pi_{i_1}(\omega), \ldots, \pi_{i_r}(\omega)) \in G\}\big) \simeq \otimes^r \lambda_0(G).$$

It is easy to show that $(\pi_{i_1}, \ldots, \pi_{i_r})$ is a measure-preserving mapping from $(\Omega, \mathcal{F}, \mathbf{P})$ to $(\times^r I, \boxtimes^r \mathcal{I}, \boxtimes^r \lambda)$. Hence, Part (2) is shown.

□

3.4.2 Proof of Theorem 3.2.3

We shall first consider a special case of Theorem 3.2.3.

Lemma 3.4.1. *If the type space $(S, \mathcal{S})$ is $[0,1]$ with the Borel σ-algebra, then there is an atomless probability space $(I, \mathcal{I}, \lambda)$ of agents such that for any given $\mathcal{I}$-measurable type function α from I to S with uniform distribution p on S, and for any given $\mathcal{S}$-measurable no-match probability function q from S to $[0,1]$,*

1. *there exists a sample space $(\Omega, \mathcal{F}, \mathbf{P})$, and a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$;*
2. *there exists an independent-in-types random partial matching π from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ to I with q as the no-match probability function.*

Proof of Lemma 3.4.1. We outline the proof first. In Step 1, we will construct the agent space $(I, \mathcal{I}, \lambda)$, the sample space $(\Omega, \mathcal{F}, \mathbf{P})$, and the random matching π. In Step 2, we will construct the type process g^α and hyperfinite type process g, where the latter is $\mathcal{I} \boxtimes \mathcal{F}$-measurable. In Step 3, we will show that g satisfies the distribution property (*i.e.*, Condition 1-(c)). In Step 4, we will show that g is essentially pairwise independent (*i.e.*, Condition 2). In Step 5, we conclude our proof by showing that g and g^α are almost same.

Step 1. Let K be any fixed unlimited hyperfinite natural number in ${}^*\mathbb{N}_\infty$. Let $I = \{1, 2, \ldots, M\}$ be the space of agents, where $M = K^2$. Let $\mathcal{I}_0$ be the collection of all the internal subsets of I, and λ_0 the internal counting probability measure on $\mathcal{I}_0$. Let $(I, \mathcal{I}, \lambda)$ be the Loeb space of the internal probability space $(I, \mathcal{I}_0, \lambda_0)$.

Let α be an $\mathcal{I}$-measurable type function from I to S with the uniform distribution $p = \lambda\alpha^{-1}$. Let $T = \{1, 2, \ldots, K\}$. Let $\mathcal{T}_0$ be the collection of all the internal subsets of T, p_0 the internal counting probability measure on $\mathcal{T}_0$, and $(T, \mathcal{T}, p')$ the Loeb space of the internal probability space $(T, \mathcal{T}_0, p_0)$.

Define $\alpha_0' \colon I \to T$ as follows: for each $k \in T$ and for any $i \in \{1, 2, \ldots, K\}$, $\alpha_0'(kK + i) = k$. Then α_0' is internal, $\lambda_0 \alpha_0'^{-1} = p_0$, and $\lambda\alpha_0'^{-1} = p'$. Let st be a map from $(T, \mathcal{T}, p')$ to $(S, \mathcal{S}, p)$, where for each $k \in T$, $\mathrm{st}(k)$ is the standard part of $\frac{k}{K}$.[9] Since st is measure

[9]For the definition and properties of standard part, see Section 1.6 in Loeb and Wolff (2000).

preserving, we have $\lambda(\operatorname{st}\circ\alpha_0')^{-1} = p$. Proposition 9.2 in Keisler (1984) implies that $(I, \mathcal{I}, \lambda)$ is homogeneous, that is, there exists an internal bijection $\sigma\colon I \to I$, such that $\alpha(i) = \operatorname{st}\circ\alpha_0' \circ \sigma(i)$ for λ-almost all $i \in I$. Let $\alpha_0 = \alpha_0' \circ \sigma$, then $\operatorname{st}\circ\alpha_0(i) = \alpha(i)$ for λ-almost all $i \in I$ (*i.e.*, α_0 is an internal lifting of α), and $\lambda_0\alpha_0^{-1} = p_0$.

For any $\mathcal{S}$-measurable no-match probability function q from S to $[0, 1]$, we will have an internal lifting $q_0\colon T \to {}^*[0, 1]$, that is, q_0 is internal and $\operatorname{st}(q_0(t)) = q(\operatorname{st}(t))$ for p'-almost all $t \in T$.[10]

For each $k \in T$, let $A_k = \alpha_0^{-1}(k)$, and $M_k = |A_k|$ with $\sum_{k=1}^{K} M_k = M$. Then $M_k/M = \lambda_0(A_k) = \lambda_0\alpha_0^{-1}(k) = p_0(k) = 1/K$, and hence for each $k \in T$, $M_k = K$ is an unlimited hyperfinite natural number.

For each $k \in T$, we will pick an internal sequence of hyperfinite natural numbers $\{m_k \mid k \in T\}$, such that $M_k - m_k \in {}^*\mathbb{N}_\infty$, and $N = \sum_{k=1}^{K}(M_k - m_k)$ is an unlimited even hyperfinite natural number. It is easy to see that

$$\frac{N}{M} = \sum_{k=1}^{K} \frac{M_k}{M}\left(1 - \frac{m_k}{M_k}\right) \simeq \sum_{k=1}^{K} p_0(k)[1 - q_0(k)].$$

For each $k \in T$, let B_k be an arbitrary internal subset of A_k with m_k elements, and let $\mathcal{P}_{m_k}(A_k)$ be the collection of all such internal subsets. For given $B_k \in \mathcal{P}_{m_k}(A_k)$ for each $k \in T$, let $\pi^{B_1,B_2,\ldots,B_K}$ be a full matching on $I - \cup_{k=1}^{K} B_k$ produced by the process described in the proof of Theorem 2.4 in Duffie and Sun (2007); there are $(N-1)!! = 1 \times 3 \times 5 \times \cdots \times (N-3) \times (N-1)$ such matchings.

Our sample space Ω is the set of all ordered tuples $(B_1, B_2, \ldots, B_K, \pi^{B_1,B_2,\ldots,B_K})$ such that $B_k \in \mathcal{P}_{m_k}(A_k)$ for each $k \in T$, and $\pi^{B_1,B_2,\ldots,B_K}$ is a full matching on $I - \cup_{k=1}^{K} B_k$. Then Ω has $[(N-1)!!] \prod_{k=1}^{K} \binom{M_k}{m_k}$ many elements in total. Let $\mathcal{F}_0$ be the collection of all the internal subsets of Ω, and $\mathbf{P}_0$ the internal counting probability measure on $\mathcal{F}_0$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the Loeb space of the internal probability space $(\Omega, \mathcal{F}_0, \mathbf{P}_0)$. Note that $(I, \mathcal{I}_0, \lambda_0)$ and $(\Omega, \mathcal{F}_0, \mathbf{P}_0)$ are atomless. Let $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ be the Loeb space of the internal product $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \lambda_0 \otimes \mathbf{P}_0)$, which is a Fubini extension of the usual product probability space.

Let J represent no-match. Define a mapping π from $I \times \Omega$ to $I \cup \{J\}$. For $i \in A_k$ and $\omega = (B_1, B_2, \ldots, B_K, \pi^{B_1,B_2,\ldots,B_K})$, if $i \in B_k$, then $\pi(i, \omega) = J$ (agent i is not matched);

[10] For the existence of internal liftings, see Chapter 5 in Loeb and Wolff (2000).

if $i \notin B_k$, then $i \in I - \cup_{k=1}^{K} B_k$, agent i is to be matched with agent $\pi^{B_1,B_2,\ldots,B_K}(i)$, and let $\pi(i,\omega) = \pi^{B_1,B_2,\ldots,B_K}(i)$. It is obvious that $\pi_\omega^{-1}(\{J\}) = \cup_{k=1}^{K} B_k$ and that the restriction of π_ω to $I - \cup_{k=1}^{K} B_k$ is a full matching on the set. Thus Condition 1-(a) in Definition 3.2.1 is satisfied.

Step 2. Let g^α be the matched type process from $I \times \Omega$ to $S \cup \{J\}$ under the type function α, that is, $g^\alpha(i,\omega) = \alpha(\pi(i,\omega))$ with $\alpha(J) = J$.

When $\sum_{k=1}^{K} p_0(k)[1 - q_0(k)]$ is infinitesimal, we know that $N/M \simeq 0$. It is clear that

$$\begin{aligned}(\lambda_0 \otimes \mathbf{P}_0)(\{(i,\omega) \mid \pi(i,\omega) \neq J\}) &= \int_\Omega \int_I 1_{\pi(i,\omega)\neq J}\, \mathrm{d}\lambda_0(i)\, \mathrm{d}\mathbf{P}_0(\omega) \\ &= \int_\Omega \frac{N}{M}\, \mathrm{d}\mathbf{P}_0(\omega) = \frac{N}{M} \simeq 0,\end{aligned}$$

and thus $(\lambda \boxtimes \mathbf{P})(\{(i,\omega) \in I \times \Omega \mid \pi(i,\omega) \neq J\}) = 0$, which means that $(\lambda \boxtimes \mathbf{P})(g^\alpha(i,\omega) \neq J) = 0$, and for λ-almost all $i \in I$, $g_i^\alpha(\omega) = J$ for $\mathbf{P}$-almost all $\omega \in \Omega$. Thus Conditions 1-(b), 1-(c) and 2 in Definition 3.2.1 are satisfied trivially; that is, one has a trivial random partial matching that is independent in types.

For the rest of the proof, assume that $\sum_{k=1}^{K} p_0(k)[1 - q_0(k)]$ is not infinitesimal. Let g_0 be the matched type process from $I \times \Omega$ to $T \cup \{J\}$, defined by $g_0(i,\omega) = \alpha_0(\pi(i,\omega))$ with $\alpha_0(J) = J$. Extending st to $T \cup \{J\}$, so that $\mathrm{st}(J) = J$, let $g\colon I \times \Omega \to S \cup \{J\}$ be $\mathrm{st}(g_0)$. Since both α_0 and π are internal, the fact that $\mathcal{I}_0 \otimes \mathcal{F}_0$ is the internal power set on $I \times \Omega$ implies that g_0 is $\mathcal{I}_0 \otimes \mathcal{F}_0$-measurable, and hence g is $\mathcal{I} \boxtimes \mathcal{F}$-measurable.[11]

Step 3. Fix an agent $i \in A_k$ for some $k \in T$. For any internal subset $\hat{C}$ in T, and for any $B_r \in \mathcal{P}_{m_r}(A_r)$, $r \in T$, let $N_{i\hat{C}}^{B_1,B_2,\ldots,B_K}$ be the number of full matchings on $\cup_{r=1}^{K}(A_r - B_r)$ such that agent i is matched to some agent in $\cup_{l\in\hat{C}}(A_l - B_l)$. It is obvious that $N_{i\hat{C}}^{B_1,B_2,\ldots,B_K}$ depends only on the numbers of the points in the sets $A_r - B_r$, $r \in T$, which are $M_r - m_r$, $r \in T$, respectively. Hence, $N_{i\hat{C}}^{B_1,B_2,\ldots,B_K}$ is independent of the particular choices of $B_1, B_2, \ldots, B_K$, and so can simply be denoted by $N_{i\hat{C}}$. Then fix any $i \in I$, we have that $\mathbf{P}_0(\{\omega \in \Omega\colon \pi_i(\omega) \in \cup_{l\in\hat{C}}(A_l - B_l)\})$ is

$$\begin{cases} |\cup_{l\in\hat{C}} (A_l - B_l)|/(N-1), & \text{if } i \notin \cup_{l\in\hat{C}}(A_l - B_l), \\ (|\cup_{l\in\hat{C}} (A_l - B_l)| - 1)/(N-1), & \text{if } i \in \cup_{l\in\hat{C}}(A_l - B_l), \end{cases}$$

[11] See Theorem 5.2.4 in Loeb and Wolff (2000).

and hence

$$\frac{N_{i\hat{C}}}{(N-1)!!} = \mathbf{P}_0(\pi_i^{-1}(\cup_{l\in\hat{C}}(A_l - B_l))) \tag{3.6}$$

$$\simeq \frac{|\cup_{l\in\hat{C}}(A_l - B_l)|}{N} = \frac{\sum_{l\in\hat{C}}(M_l - m_l)}{N}. \tag{3.7}$$

It can be checked that the internal cardinality of the event $\{\omega \in \Omega \mid g_0(i,\omega) \in \hat{C}\}$ is

$$\left|\left\{\omega \in \Omega \mid i \in A_k - B_k, \pi_i^{B_1,B_2,\ldots,B_K}(\omega) \in \cup_{l\in\hat{C}}(A_l - B_l)\right\}\right| \tag{3.8}$$

$$= \binom{M_k - 1}{m_k} \prod_{r\neq k} \binom{M_r}{m_r} N_{i\hat{C}}, \tag{3.9}$$

for $\omega = (B_1, B_2, \ldots, B_K, \pi^{B_1,B_2,\ldots,B_K})$.

Hence Eqs. (3.6) and (3.8) imply that

$$\begin{aligned}
&\mathbf{P}_0(\{\omega \in \Omega \mid g_0(i,\omega) \in \hat{C}\}) \\
&= \frac{M_k - m_k}{M_k} \frac{N_{i\hat{C}}}{(N-1)!!} \simeq [1 - q_0(k)] \frac{\sum_{l\in\hat{C}}(M_l - m_l)}{N} \\
&= [1 - q_0(\alpha_0(i))] \frac{\sum_{l\in\hat{C}}(1 - m_l/M_l)M_l/M}{N/M} \\
&\simeq [1 - q_0(\alpha_0(i))] \frac{\sum_{l\in\hat{C}} p_0(l)[1 - q_0(l)]}{\sum_{r=1}^{K} p_0(r)[1 - q_0(r)]}.
\end{aligned} \tag{3.10}$$

It is also easy to see that

$$\mathbf{P}_0(\{\omega \in \Omega \mid g_0(i,\omega) = J\}) = \frac{m_k}{M_k} \simeq q_0(k). \tag{3.11}$$

For any Loeb measure zero set $\bar{C} \subset T$, there exists a sequence of internal sets $\{\hat{C}_n\}_{n=1}^{\infty}$, such that $\bar{C} \subset \hat{C}_n$, and $p'(\hat{C}_n) \downarrow 0$. Then by Eq. (3.10), we have

$$\begin{aligned}
&\text{Loeb outer measure of } g_{0,i}^{-1}(\bar{C}) \\
&\leq \mathrm{st} \circ \mathbf{P}_0(g_{0,i}^{-1}(\hat{C}_n)) = [1 - q(\alpha(i))] \frac{\int_{\hat{C}_n} [1 - \mathrm{st} \circ q_0(l)] \, \mathrm{d}p'(l)}{\int_T [1 - \mathrm{st} \circ q_0(l)] \, \mathrm{d}p'(l)} \\
&\leq \frac{[1 - q(\alpha(i))]}{\int_T [1 - \mathrm{st} \circ q_0(l)] \, \mathrm{d}p'(l)} \int_{\hat{C}_n} 1 \, \mathrm{d}p'(l) = a \cdot p'(\hat{C}_n)
\end{aligned} \tag{3.12}$$

where $a = [1 - q(\alpha(i))] / \int_T [1 - \mathrm{st} \circ q_0(l)] \, \mathrm{d}p'(l)$ is a positive constant number. Let n go to infinity, then we have that $g_{0,i}^{-1}(\bar{C})$ is Loeb measurable with measure zero.

Then we will have

$$\mathbf{P}(\{\omega \in \Omega \mid g_0(i, \omega) \in \hat{C}\}) = [1 - q(\alpha(i))] \frac{\int_{\hat{C}} [1 - \mathrm{st} \circ q_0(k)] \, \mathrm{d}p'(k)}{\int_T [1 - \mathrm{st} \circ q_0(k)] \, \mathrm{d}p'(k)}$$

for λ-almost all $i \in I$ and any Loeb measurable set $\hat{C}$.

For any Borel measurable set $C \in \mathcal{S}$, $\mathrm{st}^{-1}(C)$ is Loeb measurable. Then the change of variables formula implies

$$\mathbf{P}(\{\omega \in \Omega \mid g(i, \omega) \in C\}) = [1 - q(\alpha(i))] \frac{\int_C [1 - q(k)] \, \mathrm{d}p(k)}{\int_S [1 - q(k)] \, \mathrm{d}p(k)}$$

for λ-almost all $i \in I$.

Similarly, we also have

$$\mathbf{P}(\{\omega \in \Omega \mid g(i, \omega) = J\}) = q(\alpha(i)),$$

for λ-almost all $i \in I$.

Hence the distribution condition on g_i is satisfied for λ-almost all $i \in I$.

Step 4. Fix agents i and $j \in I$ with $i \neq j$. We first consider the case that both i and j are in the same type $k \in T$, in the following three substeps.

Substep 4-1. Then we will have

$$\begin{aligned} \mathbf{P}_0(\{\omega \in \Omega \mid g_0(i, \omega) = J, g_0(j, \omega) = J\}) &= \frac{m_k(m_k - 1)}{M_k(M_k - 1)} \\ \simeq \left(\frac{m_k}{M_k}\right)^2 = \mathbf{P}_0(\{\omega \in \Omega \mid g_0(i, \omega) = J\}) &\times \mathbf{P}_0(\{\omega \in \Omega \mid g_0(j, \omega) = J\}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\mathbf{P}(\{\omega \in \omega \mid g(i, \omega) = J, g(j, \omega) = J\}) \\ &= \mathbf{P}(\{\omega \in \Omega \mid g(i, \omega) = J\}) \times \mathbf{P}(\{\omega \in \Omega \mid g(j, \omega) = J\}), \end{aligned}$$

for λ-almost all $i \in I$ and λ-almost all $j \in I$.

Substep 4-2. It can be checked that the internal cardinality of the event $\{\omega \in \Omega \mid g_0(i,\omega) \in \hat{C}, g_0(j,\omega) = J\}$ is

$$\begin{aligned}&\left|\left\{\omega \in \Omega \mid i \in A_k - B_k, j \in B_k, \pi_i^{B_1,B_2,\ldots,B_K}(\omega) \in \cup_{l\in\hat{C}}(A_l - B_l)\right\}\right| \\ =& \binom{M_k-2}{m_k-1}\prod_{r\neq k}\binom{M_r}{m_r}N_{i\hat{C}}, \end{aligned} \tag{3.13}$$

Hence Eqs. (3.6) and (3.13) imply that

$$\begin{aligned}&\mathbf{P}_0(\{\omega \in \Omega \mid g_0(i,\omega) \in \hat{C}, g_0(j,\omega) = J\}) \\ =& \frac{m_k(M_k-m_k)}{M_k(M_k-1)}\frac{N_{i\hat{C}}}{(N-1)!!} \\ \simeq& \frac{m_k}{M_k}\frac{M_k-m_k}{M_k}\frac{N_{i\hat{C}}}{(N-1)!!} \\ \simeq& \mathbf{P}_0(g_0(i,\omega) = J) \times \mathbf{P}_0(g_0(i,\omega) \in \hat{C}). \end{aligned} \tag{3.14}$$

For any $C \in \mathcal{S}$, by the same argument in the paragraph including Eq. (3.12) in Step 3, we have

$$\begin{aligned}&\mathbf{P}(\{\omega \in \Omega \mid g(i,\omega) \in C, g(j,\omega) = J\}) \\ =& \mathbf{P}(\{\omega \in \Omega \mid g(i,\omega) \in C\}) \times \mathbf{P}(\{\omega \in \Omega \mid g(j,\omega) = J\}),\end{aligned}$$

for λ-almost all $i \in I$ and λ-almost all $j \in I$.

Therefore the events $\{\omega \in \Omega \mid g(i,\omega) \in C\}$ and $\{\omega \in \Omega \mid g(j,\omega) = J\}$ are independent for λ-almost all $i \in I$ and λ-almost all $j \in I$. Similarly, the events $\{\omega \in \Omega \mid g(i,\omega) = J\}$ and $\{\omega \in \Omega \mid g(j,\omega) \in C\}$ are independent for λ-almost all $i \in I$ and λ-almost all $j \in I$.

Substep 4-3. For any internal subsets $\hat{C}, \hat{D}$ of T, and for any $B_k \in \mathcal{P}_{m_k}(A_k)$, $k \in T$, let $N_{il\hat{C}\hat{D}}^{B_1,B_2,\ldots,B_K}$ be the number of full matchings on $\cup_{k=1}^{K}(A_k - B_k)$ such that agents i and j are matched to some agents respectively in $\cup_{l\in\hat{C}}A_l - B_l$ and $\cup_{t\in\hat{D}}A_t - B_t$. As in the case of $N_{i\hat{C}}^{B_1,B_2,\ldots,B_K}$, $N_{il\hat{C}\hat{D}}^{B_1,B_2,\ldots,B_K}$ is independent of the particular choices of $B_1, B_2, \ldots, B_K$ and can simply be denoted by $N_{il\hat{C}\hat{D}}$. By taking $G = \cup_{l\in\hat{C}}(A_l - B_l) \times \cup_{t\in\hat{D}}(A_t - B_t)$, we have

$$\frac{N_{il\hat{C}\hat{D}}}{(N-1)!!} = \mathbf{P}_0(\{\omega \in \Omega \mid (\pi_i(\omega), \pi_j(\omega)) \in G\})$$
$$\simeq (\lambda_0 \otimes \lambda_0)(G) = \frac{\sum_{l\in\hat{C}}(M_l - m_l)}{N} \frac{\sum_{t\in\hat{D}}(M_t - m_t)}{N}. \tag{3.15}$$

The event $\{\omega \in \Omega \mid g_0(i,\omega) \in \hat{C}, g_0(j,\omega) \in \hat{D}\}$ is actually the set of all the $\omega = (B_1, B_2, \ldots, B_K, \pi^{B_1,B_2,\ldots,B_K})$ such that both i and j are in the same type k, and agents i and j are matched to some agents in $\cup_{l\in\hat{C}}(A_l - B_l)$ and $\cup_{t\in\hat{D}}(A_t - B_t)$, respectively. Thus, the internal cardinality of $\{\omega \in \Omega \mid g_0(i,\omega) \in \hat{C}, g_0(j,\omega) \in \hat{D}\}$ is

$$\binom{M_k - 2}{m_k} \prod_{r\neq k} \binom{M_r}{m_r} N_{il\hat{C}\hat{D}}. \tag{3.16}$$

Hence Eqs. (3.10), (3.15) and (3.16) imply that

$$\begin{aligned}
&\mathbf{P}_0(\{\omega \in \Omega \mid g_0(i,\omega) \in \hat{C}, g_0(j,\omega) \in \hat{D}\}) \\
&= \frac{(M_k - m_k)(M_k - m_k - 1)}{M_k(M_k - 1)} \frac{N_{il\hat{C}\hat{D}}}{(N-1)!!} \\
&\simeq \left(\frac{M_k - m_k}{M_k}\right)^2 \frac{\sum_{l\in\hat{C}}(M_l - m_l)}{N} \frac{\sum_{t\in\hat{D}}(M_t - m_t)}{N} \\
&\simeq \mathbf{P}_0(\{\omega \in \Omega \mid g_0(i,\omega) \in \hat{C}\}) \times \mathbf{P}_0(\{\omega \in \Omega \mid g_0(j,\omega) \in \hat{D}\}).
\end{aligned} \tag{3.17}$$

For any $C, D \in \mathcal{S}$, by the same argument in the paragraph including Eq. (3.12) in Step 3, we have

$$\begin{aligned}
&\mathbf{P}(\{\omega \in \Omega \mid g(i,\omega) \in C, g(j,\omega) \in D\}) \\
&= \mathbf{P}(\{\omega \in \Omega \mid g(i,\omega) \in C\}) \times \mathbf{P}(\{\omega \in \Omega \mid g(j,\omega) \in D\}),
\end{aligned}$$

for λ-almost all $i \in I$ and λ-almost all $j \in I$.

Therefore the events $\{\omega \in \Omega \mid g(i,\omega) \in C\}$ and $\{\omega \in \Omega \mid g(j,\omega) \in D\}$ are independent for λ-almost all $i \in I$ and λ-almost all $j \in I$.

Substep 4-4. For the case that agent i and agent j are in the different types, one can use computations similar to methods in the above three substeps to show that the type process g is essentially pairwise independent. The details are omitted here.

Step 5. We have proven the result for the type function α_0. We still need to prove it for α (and for $g^\alpha = \alpha \circ \pi$). For any agent $i \in A_k$, for some $k \in T$. For any internal set $A \in \mathcal{I}_0$, and for any $B_r \in \mathcal{P}_{m_r}(A_r)$, $r = 1, 2, \ldots, K$, let $N_{iA}^{B_1,B_2,\ldots,B_K}$ be the number of full matchings on $\cup_{r=1}^K (A_r - B_r)$ such that agent i is matched to some agent in $A - \cup_{r=1}^K B_r$. Then, we have

$$\frac{N_{iA}^{B_1,B_2,\ldots,B_K}}{(N-1)!!} = \mathbf{P}_0(\pi_i^{-1}(A - \cup_{r=1}^K B_r)) \simeq \lambda_0(A - \cup_{r=1}^K B_r) = \frac{|A - \cup_{r=1}^K B_r|}{N}. \tag{3.18}$$

The internal event $\pi_i^{-1}(A)$ is, for $\omega = (B_1, B_2, \ldots, B_K, \pi^{B_1,B_2,\ldots,B_K})$,

$$\left\{\omega \in \Omega \mid i \in A_k - B_k, \pi_i^{B_1,B_2,\ldots,B_K}(\omega) \in \left(A - \cup_{r=1}^K B_r\right)\right\}. \tag{3.19}$$

Hence Eqs. (3.18) and (3.19) imply that

$$\begin{aligned}
\mathbf{P}_0(\pi_i^{-1}(A)) &= \sum_{B_k \in \mathcal{P}_{m_k}(A_k - \{i\}),\ B_l \in \mathcal{P}_{m_l}(A_l) \text{ for } l \neq k} \frac{1}{\prod_{r=1}^K \binom{M_r}{m_r}} \frac{N_{iA}^{B_1,B_2,\ldots,B_K}}{(N-1)!!} \\
&\simeq \sum_{B_k \in \mathcal{P}_{m_k}(A_k - \{i\}),\ B_l \in \mathcal{P}_{m_l}(A_l) \text{ for } l \neq k} \frac{1}{\prod_{r=1}^K \binom{M_r}{m_r}} \frac{|A - \cup_{r=1}^K B_r|}{N} \\
&\leq \frac{|A|}{N} \frac{\binom{M_k - 1}{m_k} \left(\prod_{r \neq k} \binom{M_r}{m_r}\right)}{\prod_{r=1}^K \binom{M_r}{m_r}} = \frac{M_k - m_k}{M_k} \frac{|A|}{M} \frac{1}{(N/M)} \\
&\simeq \frac{[1 - q_0(k)]\lambda_0(A)}{\sum_{r=1}^K p_0(r)[1 - q_0(r)]}.
\end{aligned}$$

Let $c = 1/\sum_{r=1}^K p_0(r)[1 - q_0(r)]$. Then, for each $i \in I$ and any $A \in \mathcal{I}_0$, $\mathbf{P}(\pi_i^{-1}(A)) \leq c \cdot \lambda(A)$. Thus, Keisler's Fubini property[12] implies that $(\lambda \boxtimes \mathbf{P})(\pi^{-1}(A)) \leq c \cdot \lambda(A)$. Let $B = \{i \in I \mid \alpha(i) \neq \mathrm{st}(\alpha_0(i))\}$. We know that $\lambda(B) = 0$, $(\lambda \boxtimes \mathbf{P})(\pi^{-1}(B)) = \mathbf{P}(\pi_i^{-1}(B)) = 0$ for each $i \in I$. Since g and g^α agree on $I \times \Omega - \pi^{-1}(B)$, g^α must be $\mathcal{I} \boxtimes \mathcal{F}$-measurable. For each $i \in I$, g_i and g_i^α agree on $\Omega - \pi_i^{-1}(B)$, and hence the relevant distribution and independence conditions are also satisfied by g^α. □

[12] See Section 5.3.7 in Loeb and Wolff (2000).

Proof of Theorem 3.2.3. In the Step 1 of proof of Lemma 3.4.1, we have constructed a agent space $(I, \mathcal{I}, \lambda)$.

For any probability measure p on S, we can find a $\mathcal{L}$-measurable function $Z\colon L \to S$, such that $\eta Z^{-1} = p$, where $L = [0,1]$, $\mathcal{L}$ is the Borel σ-algebra, and η is the Lebesgue measure. Given any $\mathcal{I}$-measurable type function α from I to S with type distribution p, we can define a new type function $\tilde{\alpha}$ from I to L with uniform distribution, such that $\lambda(Z \circ \tilde{\alpha})^{-1} = \lambda\alpha^{-1}$. Proposition 9.2 in Keisler (1984) implies that $(I, \mathcal{I}, \lambda)$ is homogeneous, that is, there exists an internal bijection $\sigma\colon I \to I$, such that $Z \circ \tilde{\alpha} \circ \sigma(i) = \alpha(i)$ for λ-almost all $i \in I$. Let $\alpha' = \tilde{\alpha} \circ \sigma$, then $Z \circ \alpha'(i) = \alpha(i)$ for λ-almost all $i \in I$.

Given any $\mathcal{S}$-measurable no-match probability function q from S to $[0,1]$, we also can define a new $\mathcal{L}$-measurable no-match probability function q' from L to $[0,1]$: $q' = q \circ Z$. Based on the construction of α' and q', we will have $q' \circ \alpha'(i) = q \circ \alpha(i)$ for λ-almost all $i \in I$.

Now applying Lemma 3.4.1 again, there exists a sample space $(\Omega, \mathcal{F}, \mathbf{P})$, a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$, and an independent-in-types random partial matching π from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ to I with q' as the no-match probability function, that is, the type process $g' = \alpha' \circ \pi : (I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P}) \to L \cup \{J\}$ satisfies the Conditions 1-(b), 1-(c), and 2 in Definition 3.2.1.

Since $g_i(\omega) = \alpha \circ \pi(i,\omega) = Z \circ \alpha' \circ \pi(i,\omega) = Z \circ g'(i,\omega) = Z \circ g'_i(\omega)$, and

$$\begin{aligned}
&\mathbf{P}(\{\omega \in \Omega \mid g'(i,\omega) \in Z^{-1}(C)\}) \\
&= \mathbf{P}(g'_i)^{-1}(Z^{-1}(C)) = \mathbf{P}(Z \circ g'_i)^{-1}(C) = \mathbf{P}(g_i)^{-1}(C) \\
&= \mathbf{P}(\{\omega \in \Omega \mid g(i,\omega) \in C\}),
\end{aligned}$$

holds for any $C \in \mathcal{S}$, Conditions 1-(b), 1-(c) and 2 are also satisfied for the type process $g = \alpha \circ \pi$. □

3.4.3 Proof of Theorem 3.2.4

Let $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability spaces constructed in the proof of Theorem 3.2.3. The following lemma is a strengthened version of Lemma 2 in Kakutani (1944) and its proof is given in Duffie and Sun (2012); see also Lemma 419I in Fremlin (2006) and Lemma 3 in Sun and Zhang (2009).

Lemma 3.4.2. *There is a disjoint family $\mathcal{C} = \{C_i : i \in I\}$ of subsets of $L = [0,1]$ such that $\cup_{i \in I} C_i = L$, and for each $i \in I$, C_i has the cardinality of the continuum, $\eta_*(C_i) = 0$ and $\eta^*(C_i) = 1$, where η_* and η^* are, respectively, the inner and outer measures of the Lebesgue measure η.*

As in the Appendix of Sun and Zhang (2009), we follow some constructions used in the proof of Lemma 521P(b) of Fremlin (2008).

Step 1 [Construction of subset C]. Define a subset C of $L \times I$ by letting $C = \{(l,i) \in L \times I : l \in C_i, i \in I\}$. Let $(L \times I, \mathcal{L} \otimes \mathcal{I}, \eta \otimes \lambda)$ be the usual product probability space. For any $\mathcal{L} \otimes \mathcal{I}$-measurable set U that contains C, $C_i \subseteq U_i$ for each $i \in I$, where $U_i = \{l \in L : (l,i) \in U\}$ is the i-section of U. The Fubini property of $\eta \otimes \lambda$ implies that for λ-almost all $i \in I$, U_i is $\mathcal{L}$-measurable, which means that $\eta(U_i) = 1$ (since $\eta^*(C_i) = 1$). Since $\eta \otimes \lambda(U) = \int_I \eta(U_i)\, \mathrm{d}\lambda$, we have $\eta \otimes \lambda(U) = 1$. Therefore, the $\eta \otimes \lambda$-outer measure of C is one.

Step 2 [Probability structure on C]. Since the $\eta \otimes \lambda$-outer measure of C is one, the method in Doob (1953) (see p. 69 therein) can be used to extend $\eta \otimes \lambda$ to a measure γ on the σ-algebra $\mathcal{U}$ generated by the set C and the sets in $\mathcal{L} \otimes \mathcal{I}$ with $\gamma(C) = 1$. It is easy to see that $\mathcal{U} = \{(U^1 \cap C) \cup (U^2 \backslash C) : U^1, U^2 \in \mathcal{L} \otimes \mathcal{I}\}$, and that $\gamma[(U^1 \cap C) \cup (U^2 \backslash C)] = \eta \otimes \lambda(U^1)$ for any measurable sets $U^1, U^2 \in \mathcal{L} \otimes \mathcal{I}$. Let $\mathcal{T}$ be the σ-algebra $\{U \cap C : U \in \mathcal{L} \otimes \mathcal{I}\}$, which is the collection of all the measurable subsets of C in $\mathcal{U}$. The restriction of γ to $(C, \mathcal{T})$ is still denoted by γ. Then, $\gamma(U \cap C) = \eta \otimes \lambda(U)$, for every measurable set $U \in \mathcal{L} \otimes \mathcal{I}$. Note that $(L \times I, \mathcal{U}, \gamma)$ is an extension of $(L \times I, \mathcal{L} \otimes \mathcal{I}, \eta \otimes \lambda)$.

Step 3 [New probability structure on L]. Consider the projection mapping $p^L : L \times I \to L$ with $p^L(l,i) = l$. Let ψ be the restriction of p^L to C. Since the family $\mathcal{C}$ is a partition of $L = [0,1]$, ψ is a bijection between C and L. It is obvious that p^L is a measure-preserving mapping from $(L \times I, \mathcal{L} \otimes \mathcal{I}, \eta \otimes \lambda)$ to $(L, \mathcal{L}, \eta)$ in the sense that for any $B \in \mathcal{L}$, $(p^L)^{-1}(B) \in \mathcal{L} \otimes \mathcal{I}$ and $\eta \otimes \lambda[(p^L)^{-1}(B)] = \eta(B)$; and thus p^L is a measure-preserving mapping from $(L \times I, \mathcal{U}, \gamma)$ to $(L, \mathcal{L}, \eta)$. Since $\gamma(C) = 1$, ψ is a measure-preserving mapping from $(C, \mathcal{T}, \gamma)$ to $(L, \mathcal{L}, \eta)$, that is, $\gamma[\psi^{-1}(B)] = \eta(B)$ for any $B \in \mathcal{L}$.

To introduce one more measure structure on the unit interval $[0,1]$, we shall denote it by $\hat{I}$. Let $\hat{\mathcal{I}}$ be the σ-algebra $\{S \subseteq \hat{I} : \psi^{-1}(S) \in \mathcal{T}\}$. Define a set function $\hat{\lambda}$ on $\hat{\mathcal{I}}$

by letting $\hat{\lambda}(S) = \gamma[\psi^{-1}(S)]$ for each $S \in \hat{\mathcal{I}}$. Since ψ is a bijection, $\hat{\lambda}$ is a well-defined probability measure on $(I, \mathcal{I})$. Moreover, ψ is also an isomorphism from $(C, \mathcal{T}, \gamma)$ to $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$. Since ψ is a measure-preserving mapping from $(C, \mathcal{T}, \gamma)$ to $(L, \mathcal{L}, \eta)$, it is obvious that $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$ is an extension of the Lebesgue unit interval $(L, \mathcal{L}, \eta)$.[13]

Step 4 [Fubini extension]. We shall now follow the procedure used in the proof of Proposition 2 in Sun and Zhang (2009) to construct a Fubini extension based on the probability spaces $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$ as defined above, and $(\Omega, \mathcal{F}, \mathbf{P})$ as in our Theorem 3.2.3 here.

First, consider the usual product space $(L \times I \times \Omega, \mathcal{L} \otimes (\mathcal{I} \boxtimes \mathcal{F}), \eta \otimes (\lambda \boxtimes \mathbf{P}))$ of the Lebesgue unit interval $(L, \mathcal{L}, \eta)$ with the Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$. In Step 1 of the proof of Proposition 2 in Sun and Zhang (2009), it is showed that the probability space $(L \times I \times \Omega, \mathcal{L} \otimes (\mathcal{I} \boxtimes \mathcal{F}), \eta \otimes (\lambda \boxtimes \mathbf{P}))$ is a Fubini extension of the usual triple product space $((L \times I) \times \Omega, (\mathcal{L} \otimes \mathcal{I}) \otimes \mathcal{F}, (\eta \otimes \lambda) \otimes \mathbf{P})$.

Next, as shown in Step 2 of the proof of Proposition 2 in Sun and Zhang (2009), the set $C \times \Omega$ has $\eta \otimes (\lambda \boxtimes \mathbf{P})$-outer measure one. Based on the Fubini extension $(L \times I \times \Omega, \mathcal{L} \otimes (\mathcal{I} \boxtimes \mathcal{F}), \eta \otimes (\lambda \boxtimes \mathbf{P}))$, we can construct a measure structure on $C \times \Omega$ as follows. Let $\mathcal{E} = \{D \cap (C \times \Omega) \colon D \in \mathcal{L} \otimes (\mathcal{I} \boxtimes \mathcal{F})\}$ (which is a σ-algebra on $C \times \Omega$), and τ be the set function on $\mathcal{E}$ defined by $\tau(D \cap (C \times \Omega)) = \eta \otimes (\lambda \boxtimes \mathbf{P})(D)$ for any measurable set D in $\mathcal{L} \otimes (\mathcal{I} \boxtimes \mathcal{F})$.[14] Then, τ is a well-defined probability measure on $(C \times \Omega, \mathcal{E})$ since the $\eta \otimes (\lambda \boxtimes \mathbf{P})$-outer measure of $C \times \Omega$ is one. In Step 2 of the proof of Proposition 2 in Sun and Zhang (2009), it is showed that the probability space $(C \times \Omega, \mathcal{E}, \tau)$ is a Fubini extension of the usual product probability space $(C \times \Omega, \mathcal{T} \otimes \mathcal{F}, \gamma \otimes \mathbf{P})$.

Let Ψ be the mapping $(\psi, \mathrm{Id}_\Omega)$ from $C \times \Omega$ to $\hat{I} \times \Omega$, where Id_Ω is the identity map on Ω. It clearly means that for each $(l, i) \in C$, $\omega \in \Omega$, $\Psi((l, i), \omega) = (\psi, \mathrm{Id}_\Omega)((l, i), \omega) = (\psi(l, i), \omega)$. Since ψ is a bijection from C to $\hat{I}$, Ψ is a bijection from $C \times \Omega$ to $\hat{I} \times \Omega$. Let $\mathcal{W} = \{H \subseteq \hat{I} \times \Omega \colon \Psi^{-1}(H) \in \mathcal{E}\}$; then $\mathcal{W}$ is a σ-algebra of subsets of $\hat{I} \times \Omega$. Define a probability measure ρ on $\mathcal{W}$ by letting $\rho(H) = \tau[\Psi^{-1}(H)]$ for any $H \in \mathcal{W}$. Therefore, Ψ is an isomorphism from the probability space $(C \times \Omega, \mathcal{E}, \tau)$ to the probability space

[13] Note that Kakutani (1944) is the first to consider the rich measure-theoretic extension of the Lebesgue unit interval. For various Lebesgue extensions, see Duffie and Sun (2012), Khan and Zhang (2012a), Podczeck (2008) and Sun and Zhang (2009). Note that, in Duffie and Sun (2012) and Sun and Zhang (2009), it is not an issue to construct a Lebesgue extension but an independent random matching with this Lebesgue extension as the agent space or a rich Fubini extension.

[14] Note that we replace the corresponding notation ν used in the Appendix of Sun and Zhang (2009) by τ in this paper. The reason is that the notation ν has been used earlier for the match-induced type-change probabilities.

$(\hat{I} \times \Omega, \mathcal{W}, \rho)$. In Step 3 of the proof of Proposition 2 in Sun and Zhang (2009), it is showed that the probability space $(\hat{I} \times \Omega, \mathcal{W}, \rho)$ is a Fubini extension of the usual product probability space $(\hat{I} \times \Omega, \hat{\mathcal{I}} \otimes \mathcal{F}, \hat{\lambda} \otimes \mathbf{P})$.

Since $(\hat{I} \times \Omega, \mathcal{W}, \rho)$ is a Fubini extension, we shall follow Definition 2.1.1 to denote $(\hat{I} \times \Omega, \mathcal{W}, \rho)$ by $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$.

Now, define a mapping φ from $\hat{I}$ to I by letting $\varphi(\hat{i}) = i$ if $\hat{i} \in C_i$. Since the family $\mathcal{C} = \{C_i : i \in I\}$ is a partition of $\hat{I} = [0, 1]$, φ is well-defined.

The following lemma is shown by Duffie and Sun in Duffie and Sun (2012).

Lemma 3.4.3 (Lemma 11 in Duffie and Sun (2012)). *The following properties of φ hold.*

1. *The mapping φ is measure preserving from $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$ to $(I, \mathcal{I}, \lambda)$, in the sense that for any $A \in \mathcal{I}$, $\varphi^{-1}(A)$ is measurable in $\hat{\mathcal{I}}$ with $\hat{\lambda}[\varphi^{-1}(A)] = \lambda(A)$.*

2. *Let Φ be the mapping $(\varphi, \mathrm{Id}_\Omega)$ from $\hat{I} \times \Omega$ to $I \times \Omega$, that is, $\Phi(\hat{i}, \omega) = (\varphi, \mathrm{Id}_\Omega)(\hat{i}, \omega) = (\varphi(\hat{i}), \omega)$ for any $(\hat{i}, \omega) \in \hat{I} \times \Omega$. Then Φ is measure preserving from $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes \mathbf{P})$ to $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ in the sense that for any $V \in \mathcal{I} \boxtimes \mathcal{F}$, $\Phi^{-1}(V)$ is measurable in $\hat{\mathcal{I}} \boxtimes \mathcal{F}$ with $(\hat{\lambda} \boxtimes \mathbf{P})[\Phi^{-1}(V)] = (\lambda \boxtimes \mathbf{P})(V)$.*

Proof of Theorem 3.2.4. It follows from Theorem 3.2.3 that for any type distribution p on $(S, \mathcal{S})$ and $\mathcal{S}$-measurable no-match probability function q from S to $[0, 1]$, there exist

1. an atomless probability space $(I, \mathcal{I}, \lambda)$ of agents, where the index space I has cardinality of the continuum;

2. a sample space $(\Omega, \mathcal{F}, \mathbf{P})$;

3. a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ on which is defined an independent random partial matching π from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P})$ to I with type distribution p and with q as the no-match probability function.

Let $\hat{\alpha}$ be the mapping $\alpha(\varphi)$ from $\hat{I}$ to S. By the measure preserving property of φ, we know that $\hat{\alpha}$ is $\hat{\mathcal{I}}$-measurable type function with distribution p on S. By the definitions of $\hat{\alpha}$, it is obvious that for each $\hat{i} \in \hat{I}$,

$$\hat{\alpha}_{\hat{i}}(\cdot) = \alpha_{\varphi(\hat{i})}(\cdot). \tag{3.20}$$

We shall first fix some bijections between the i-sections of the set C. For any $i, i' \in I$ with $i \neq i'$, let $\Theta^{i,i'}$ be a bijection from C_i to $C_{i'}$, and $\Theta^{i',i}$ be the inverse mapping of $\Theta^{i,i'}$. This is possible since both C_i and $C_{i'}$ have cardinality of the continuum, as noted in Lemma 3.4.2.

Define a mapping $\hat{\pi}$ from $\hat{I} \times \Omega$ to $\hat{I} \cup \{J\}$ such that for each $(\hat{i}, \omega) \in \hat{I} \times \Omega$,

$$\hat{\pi}(\hat{i}, \omega) = \begin{cases} J, & \text{if } \pi_\omega(\varphi(\hat{i})) = J; \\ \Theta^{\varphi(\hat{i}), \pi_\omega(\varphi(\hat{i}))}(\hat{i}), & \text{if } \pi_\omega(\varphi(\hat{i})) \neq J. \end{cases}$$

When $\pi_\omega(\varphi(\hat{i})) \neq J$, π_ω defines a full matching on $I - \pi_\omega^{-1}(\{J\})$, which implies that $\pi_\omega(\varphi(\hat{i})) \neq \varphi(\hat{i})$. Hence, $\hat{\pi}$ is a well-defined mapping from $\hat{I} \times \Omega$ to $\hat{I} \cup \{J\}$.

Next we consider the partial matching properties of $\hat{\pi}$. By the similar methods adopted in the proof of Theorem 4 in Duffie and Sun (2012), we will have

1. $\hat{\pi}_\omega$ is a full matching on $\hat{I} - \hat{\pi}_\omega(\{J\})$.
2. Extending $\hat{\alpha}$ so that $\hat{\alpha}(J, \omega) = J$ for any $\omega \in \Omega$, we define $\hat{g} \colon \hat{I} \times \Omega \to S \cup \{J\}$ by $\hat{g} = \hat{\alpha} \circ \hat{\pi}$. Define $\varphi(J) = J$. Then $\hat{g}$ is $\hat{\mathcal{I}} \boxtimes \mathcal{F}$-measurable and for each $\hat{i} \in \hat{I}$,
$$\hat{g}_{\hat{i}}(\cdot) = g_{\varphi(\hat{i})}(\cdot). \tag{3.21}$$
3. Eqs. (3.1), (3.20) and (3.21) imply that for $\hat{\lambda}$-almost all $\hat{i} \in \hat{I}$,
$$\mathbf{P}(\hat{g}_{\hat{i}} = J) = \mathbf{P}(g_{\varphi(\hat{i})} = J) = q(\alpha(\varphi(\hat{i}))) = q(\hat{\alpha}(\hat{i})),$$
and
$$\begin{aligned}\mathbf{P}(\hat{g}_{\hat{i}} \in C) = \mathbf{P}(g_{\varphi(\hat{i})} \in C) &= [1 - q(\alpha(\varphi(\hat{i})))] \frac{\int_C [1 - q(k)] \, \mathrm{d}p(k)}{\int_S [1 - q(k)] \, \mathrm{d}p(k)} \\ &= [1 - q(\hat{\alpha}(\hat{i}))] \frac{\int_C [1 - q(k)] \, \mathrm{d}p(k)}{\int_S [1 - q(k)] \, \mathrm{d}p(k)},\end{aligned}$$
for any $C \in \mathcal{S}$.

By Eq. (3.21), the essential pairwise independence of $\hat{g}$ follows immediately from that of g and the measure-preserving property of φ. Therefore, we have constructed an new independent-in-types random partial matching $\hat{\pi}$ from $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes \mathbf{P})$ to $\hat{I}$, with the given type distribution p and no-match probability function q. $\square$

3.4.4 Proof of Proposition 3.3.1

Proof of Proposition 3.3.1. If $p(B) = 0$, Eqs. (3.2) and (3.3) are automatically satisfied. Consider only $p(B) > 0$.

Then by Fact 2.1.3, for $\mathbf{P}$-almost all $\omega \in \Omega$, the sample function $g_\omega^{\alpha^{-1}(B)}$ on $\alpha^{-1}(B)$ has the same distribution as $g^{\alpha^{-1}(B)}$ on $\alpha^{-1}(B) \times \Omega$. Hence, for $\mathbf{P}$-almost all $\omega \in \Omega$,

$$\lambda^{\alpha^{-1}(B)} \left(\left(g_\omega^{\alpha^{-1}(B)} \right)^{-1} (\{J\}) \right) = (\lambda^{\alpha^{-1}(B)} \boxtimes \mathbf{P}) \left(\left(g^{\alpha^{-1}(B)} \right)^{-1} (\{J\}) \right),$$

which means that

$$\begin{aligned} \lambda(\{i \in I \mid \alpha(i) \in B, g_\omega(i) = J\}) &= \int_{\alpha^{-1}(B)} \int_\Omega 1_{(g_i = J)} \, \mathrm{d}\mathbf{P}(\omega) \, \mathrm{d}\lambda(i) \\ &= \int_{\alpha^{-1}(B)} q(\alpha(i)) \, \mathrm{d}\lambda(i) = \int_B q(k) \, \mathrm{d}p(k), \end{aligned}$$

where 1_C denotes its indicator function for a set C in a space, and the last equation is a corollary of classical change of variables formula.

Similarly, for any $C \in \mathcal{S}$, for $\mathbf{P}$-almost all $\omega \in \Omega$, we have

$$\lambda(\alpha^{-1}(B) \cap g_\omega^{-1}(C)) = (\lambda \boxtimes \mathbf{P})\big((\alpha^{-1}(B) \times \Omega) \cap g^{-1}(C)\big),$$

which means that

$$\begin{aligned} \lambda(\{i \in I \mid \alpha(i) \in B, g_\omega(i) \in C\}) &= \int_{\alpha^{-1}(B)} \int_\Omega 1_{(g_i \in C)} \, \mathrm{d}\mathbf{P}(\omega) \, \mathrm{d}\lambda(i) \\ &= \int_{\alpha^{-1}(B)} [1 - q(\alpha(i))] \frac{\int_C [1 - q(l)] \, \mathrm{d}p(l)}{\int_S [1 - q(l)] \, \mathrm{d}p(l)} \, \mathrm{d}\lambda(i) \\ &= \int_B [1 - q(k)] \, \mathrm{d}p(k) \frac{\int_C [1 - q(l)] \, \mathrm{d}p(l)}{\int_S [1 - q(l)] \, \mathrm{d}p(l)}. \end{aligned}$$

$\square$

Chapter 4

Nonatomic games with infinite-dimensional action spaces

4.1 Introduction

It is common sense that pure-strategy Nash equilibria may not exist in general non-cooperative games. However, it is important from a game-theoretical point of view to know when pure-strategy Nash equilibria exist. For a finite-player game, the existence of pure-strategy Nash equilibria follows from certain conditions on the payoff functions and strategy spaces.[1] For games with a nonatomic measure-theoretic structure that models the space of players or information, a general purification principle due to Dvoretsky, Wald and Wolfowitz (1951a) guarantees that one can always obtain a pure-strategy Nash equilibrium from a mixed-strategy Nash equilibrium, when the action space is finite; see Dvoretsky, Wald and Wolfowitz (1951b), Khan, Rath and Sun (2006) and their references. For games with countable actions, similar results on pure-strategy Nash equilibria can be found in Khan and Sun (1995).

For games with a nonatomic measure-theoretical structure and an uncountable compact metric action space, when the players' payoffs depend on their own actions and the action distribution of other players, there are several subtle possibilities. First, when the space of players or information is modeled by the Lebesgue unit interval, counterex-

[1]The supermodular game is a good example; see Topkis (1979). Here is another example, when each player's payoff function is quasi-concave and continuous, and the strategy space is a convex and compact subset of a finite-dimensional Euclidean space, there exists a pure-strategy Nash equilibrium by Kakutani's fixed-point theorem.

amples are constructed to show the nonexistence of pure-strategy Nash equilibria; see Khan, Rath and Sun (1997, 1999). Second, when the Lebesgue unit interval is replaced by a nonatomic Loeb space, positive results on pure-strategy Nash equilibria are shown in Khan and Sun (1999). Third, for a fixed nonatomic player space, it is shown in Keisler and Sun (2009) that any game with the given player space has a pure-strategy Nash equilibrium if and only if the underlying player space is saturated in the sense that any subspace is not countably generated modulo the null sets.[2]

The purpose of this chapter is to consider the pure-strategy Nash equilibria for games with a nonatomic player space and an uncountable compact action set in an infinite-dimensional Banach space, where players' payoffs depend on their own actions and the average action of other players.[3] As shown in Khan, Rath and Sun (1997), when the player space is the Lebesgue unit interval and the action space is an uncountable compact subset of the Hilbert space ℓ_2—the space of square-summable real-valued sequences, pure-strategy Nash equilibria may not exist. Since various infinite-dimensional Banach spaces are widely used in the economics literature,[4] a natural question is whether we could find a right infinite-dimensional Banach space rather than ℓ_2 to deliver a positive result on the existence of the pure-strategy Nash equilibria. We show that this is impossible as long as the player space is the Lebesgue unit interval. In particular, given any infinite-dimensional Banach space, there always exist nonatomic games with an uncountable compact action set in this Banach space such that these games do not have pure-strategy Nash equilibria, provided that the player space is the Lebesgue unit interval.

Nevertheless, if the player space is not the Lebesgue unit interval, it is possible to deliver a positive result on pure-strategy Nash equilibria for nonatomic games with infinite-dimensional action spaces. Khan and Sun (1999) show that when the Lebesgue unit interval is replaced by a nonatomic Loeb space, there exists a pure-strategy Nash equilibrium for any nonatomic game with any uncountable compact action set in an

[2]Similar results hold for finite-player games with nonatomic information spaces, see Khan and Zhang (2012b). See Fu (2007) for the relation between the games with nonatomic player spaces and finite-player games with nonatomic information spaces.

[3]When players' payoff functions are linear in their own actions, see Khan (1986) for the existence of the approximate pure-strategy Nash equilibria for nonatomic games with infinite-dimensional action spaces; see also Rustichini and Yannelis (1991) for the existence of exact pure-strategy Nash equilibria for nonatomic games with infinite-dimensional action spaces, provided that the game has "many more" players than actions.

[4]See, for example, Bewley (1972), Yannelis (2009) and the books Khan and Yannelis (1991) and Stokey, Lucas and Prescott (1989).

infinite-dimensional Banach space.[5] It follows from the existence result in Khan and Sun (1999) and general saturation property that the existence result of pure-strategy Nash equilibria still holds when the player space is modeled by a saturated probability space.

A further and more interesting question is whether the converse of the above result is true. In other words, whether the necessity result of the saturation property in Keisler and Sun (2009) can be established in the context of nonatomic games with infinite-dimensional action spaces. We provide an answer in the affirmative. In particular, we show that given a nonatomic player space and a fixed compact subset of a fixed infinite-dimensional Banach space, if every game with this compact subset as the common action set has a pure-strategy Nash equilibrium, then the underlying player space must be a saturated probability space; see Theorem 4.4.7. Put differently, if the player space is not a saturated probability space, then one can always construct a nonatomic game with this player space where players take actions from a given infinite-dimensional Banach space, such that it has no pure-strategy Nash equilibrium. It is worthwhile to note that our necessity result is not implied by the necessity part of Theorem 4.6 in Keisler and Sun (2009).

To summarize, to obtain a positive result on the existence of pure-strategy Nash equilibria for nonatomic games with actions in infinite-dimensional spaces, the measure-theoretic structure of the player space plays a fundamental role. It is worth noting that as far as the above counterexamples on Lebesgue interval are concerned, to guarantee the existence of pure-strategy Nash equilibria, one is not necessary to turn to saturated probability spaces, a simple extension of the Lebesgue unit interval does serve this purpose; see Proposition 4.5.1 below.

This chapter is organized as follows. Section 4.2 presents basic definitions and notations on the Banach spaces and on the nonatomic games. In Section 4.3, nonatomic games are constructed on the Lebesgue unit interval with actions in any infinite-dimensional Banach space, such that there does not have a pure-strategy Nash equilibrium for each of them. The sufficiency and necessity results are established in Section 4.4. We conclude in Section 4.5. The technical proofs are collected in Section 4.6.

[5]For nonatomic games with a compact action set in a finite-dimensional space, the existence of pure-strategy Nash equilibria is shown in Rath (1992). See also Khan *et al.* (1997) for the positive results on pure-strategy Nash equilibria for games with countable actions in an infinite-dimensional space.

4.2 Basics

Let $(T, \mathcal{T}, \mu)$ be a nonatomic probability space. In this chapter, the Lebesgue unit interval is denoted by $(L, \mathcal{L}, \eta)$, that is, the unit interval $L = [0, 1]$ is endowed with the Lebesgue σ-algebra $\mathcal{L}$ and the Lebesgue measure η. Moreover, $\mathbb{N}$ denotes the set of all nonnegative integers.

Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with the norm $\|\cdot\|$. Denote by $\mathrm{d}(\cdot, \cdot)$ the distance operator on X defined as $\mathrm{d}(x, y) = \|x - y\|$ for any $x, y \in X$. Let X^* be the dual space of $(X, \|\cdot\|)$. It is a well-known result that there exists a biorthogonal system on $X^* \times X$, denoted by $\{(x_n^*, x_n)\colon\ x_n^* \in X^*, x_n \in X \text{ and } n \in \mathbb{N}\}$, such that $x_m^*(x_n) = 0$ for any distinct $m, n \in \mathbb{N}$ and $x_n^*(x_n) = 1$ for all $n \in \mathbb{N}$.[6]

We next review some concepts and notations on the measurability and integrability for functions from $(T, \mathcal{T}, \mu)$ to $(X, \|\cdot\|)$.[7] A function $f\colon T \to X$ is said to be **$\mathcal{T}$-measurable** if there exists a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0$ for μ-almost all $t \in T$. A $\mathcal{T}$-measurable function f is called **Bochner integrable** if there is a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \int_T \|f_n - f\| \,\mathrm{d}\mu = 0$, the Bochner integral of f, denoted by $\int_T f \,\mathrm{d}\mu$, is defined to be $\lim_{n\to\infty} \int_T f_n \,\mathrm{d}\mu$; similarly, $\int_S f \,\mathrm{d}\mu$ is defined to be $\lim_{n\to\infty} \int_S f_n \,\mathrm{d}\mu$ for any $S \in \mathcal{T}$. Let $F\colon T \twoheadrightarrow X$ be a correspondence from $(T, \mathcal{T}, \mu)$ to X. A **selection** of F is a $\mathcal{T}$-measurable function f such that $f(t) \in F(t)$ for all $t \in T$. The **Bochner integral of** F is defined as follows,

$$\int_T F \,\mathrm{d}\mu = \left\{ \int_T f \,\mathrm{d}\mu\colon f \text{ is a Bochner integrable selection of } F \right\}.$$

Let A be a nonempty norm-compact (henceforth compact) subset of X, and $\overline{\mathrm{con}}(A)$ the closed convex hull of A. Note that $\overline{\mathrm{con}}(A)$ is also a compact set.[8] Moreover,

$$\overline{\mathrm{con}}(A) \supseteq \left\{ \int_T f \,\mathrm{d}\eta(t)\colon f \text{ is a Bochner integrable function from } T \text{ to } A \right\}. \tag{4.1}$$

Let $\mathcal{U}_A$ be the space of norm-continuous (henceforth continuous) real-valued functions on $A \times \overline{\mathrm{con}}(A)$ endowed with the sup-norm topology and its corresponding Borel σ-algebra which is generated by this topology.

6 See Proposition 1.f.3 in Lindenstrauss and Tzafriri (1977).
7 For more details, see Chapter 2 of Diestel and Uhl (1977).
8 This is Mazur's Theorem; see Theorem 12 in Diestel and Uhl (1977).

Finally, we specify the model of the nonatomic games. The player space is modeled by a nonatomic probability space $(T, \mathcal{T}, \mu)$. It is worthwhile to note that we use the phrase "player space" instead of "the set of players" or "the set of players' names" because we would like to emphasize the essence of the underlying probability structure in this paper. A nonempty compact set $A \subseteq X$ will be the common action set available for all the players, and the closed convex hull of $\overline{\text{con}}(A)$ serves as the space of all the possible societal responses. The payoff function for any player is a continuous function on $A \times \overline{\text{con}}(A)$, *i.e.*, an element in $\mathcal{U}_A$. Put differently, the payoff of every player continuously depends on this player's own action and the "average" of the actions of others players. Suppose that the action profile is a $\mathcal{T}$-measurable function $f: T \to A$, the nonatomic condition allows us to model the average of others players by the Bochner integral $\int_T f \, \mathrm{d}\mu$. It follows from (4.1) that $\int_T f \, \mathrm{d}\mu \in \overline{\text{con}}(A)$. Now we are ready to present the definitions of nonatomic games and pure-strategy Nash equilibria.

Definition 4.2.1. *A* ***nonatomic game*** *with the player space* $(T, \mathcal{T}, \mu)$ *and the common action set* $A \subseteq (X, \|\cdot\|)$ *is a* $\mathcal{T}$*-measurable function* $\mathcal{G}$ *from* T *to* $\mathcal{U}_A$*. A* $\mathcal{T}$*-measurable function* $f: T \to A$ *is called a* ***pure-strategy Nash equilibrium of*** $\mathcal{G}$ *if for* μ*-almost all* $t \in T$ *and all* $a \in A$*,*

$$\mathcal{G}(t)\left(f(t), \int_T f(t) \, \mathrm{d}\mu(t)\right) \geq \mathcal{G}(t)\left(a, \int_T f(t) \, \mathrm{d}\mu(t)\right).$$

4.3 Counterexamples

In this section, we will present a class of nonatomic games $\mathcal{G}_s$ $s \in (0, 1]$, each of which does not have a pure-strategy Nash equilibrium. In these games, the player space is the Lebesgue unit interval $(L, \mathcal{L}, \eta)$ and players can take actions from any infinite-dimensional Banach space.

4.3.1 Preliminaries

We first review the Walsh system $\{W_n\}_{n \in \mathbb{N}}$ defined on the Lebesgue unit interval $(L, \mathcal{L}, \eta)$. Here $W_0 \equiv 1$. For any $n \geq 1$, let the binary representation be $n = n_0 + 2n_1 + \cdots + 2^{m-1} n_{m-1}$, where n_k is either 0 or 1 for $0 \leq k \leq m-2$ and $n_{m-1} = 1$. The n-th Walsh function W_n is defined as follows: for any $t \in [0,1]$, denote the binary representation[9] by $t = \frac{t_0}{2} + \frac{t_1}{2^2} + \cdots + \frac{t_{k-1}}{2^k} + \cdots$, where each t_k is either 0 or 1,

$$W_n(t) = (-1)^{n_0 t_0 + n_1 t_1 + \cdots + n_{m-1} t_{m-1}}.$$

It is well-known that $\{W_n(t)\}_{n \in \mathbb{N}}$ is a complete orthogonal basis of the space of square-integrable functions on the Lebesgue unit interval.[10] For each $n \in \mathbb{N}$, let $E_n = \{t \in L\colon W_n(t) = 1\}$. It is clear that $E_0 = L$ and E_n is a union of some subintervals in L. Moreover, for each integer $n \geq 1$, $\int_L W_0 \cdot W_n \,\mathrm{d}\eta = 0$, so we have that $\eta(E_n) = 1/2$.

Let $(X, \|\cdot\|)$ be an arbitrary infinite-dimensional Banach space associated with the fixed biorthogonal system $\{(x_n^*, x_n)\colon n \in \mathbb{N}\}$ in Section 4.2. Define a function $\psi\colon L \to X$ as follows, for any $t \in L$,

$$\psi(t) = \sum_{n=0}^{\infty} \frac{x_n}{2^n \|x_n\|} W_n(t). \tag{4.2}$$

It is easy to see that ψ is a Bochner integrable function.[11] Let $e \in X$ be the value of Bochner integral of ψ, it is clear that

$$e = \int_L \psi(t) \,\mathrm{d}\eta(t) = \frac{x_0}{\|x_0\|}. \tag{4.3}$$

Moreover, let $\Psi\colon L \twoheadrightarrow X$ be the correspondence defined as follows, for any $t \in L$,

$$\Psi(t) = \{0, \psi(t)\}. \tag{4.4}$$

Note that both 0 and e are in the Bochner integral of the correspondence Ψ. However, we have the following negative result, which plays a central rule in our construction. The proof is given in Section 4.6.

Lemma 4.3.1. $\frac{e}{2} \notin \int_L \Psi \,\mathrm{d}\eta.$

Remark 4.3.2. Let $G\colon \mathcal{L} \to X$ be a vector measure defined as $G(E) = \int_E \psi \,\mathrm{d}\eta$ for any $E \in \mathcal{L}$. It is clear that the range of the vector measure, $G(\mathcal{L}) = \{G(E)\colon E \in \mathcal{L}\}$, is identical to $\int_L \Psi \,\mathrm{d}\eta$. One implication of Lemma 4.3.1 is that the Bochner integral of

[9] In the case that a number has two binary representations, *e.g.*, $\frac{1}{2}$ has two binary representation, either $\frac{1}{2}$ itself or $\frac{1}{2^2} + \frac{1}{2^3} + \cdots$, we choose and fix the simpler representation.

[10] See Walsh (1923).

[11] Take $s_m(t) = \sum_{n=0}^{m} \frac{x_n}{2^n \|x_n\|} W_n(t)$. Then $\{s_m(t)\}_{m \in \mathbb{N}}$ is a sequence of Bochner integrable simple functions, and $\lim_{m \to \infty} \int_0^1 \|\psi(t) - s_m(t)\| \,\mathrm{d}\eta(t) = 0$.

Ψ, or the range of the vector measure $G(\mathcal{L})$, is not a convex set. The classic Lyapunov example on the range of vector measures (see p. 262 of Diestel and Uhl (1977)) is a special case of Lemma 4.3.1 where $X = \ell_2$, the space of square-summable sequences, endowed with the norm $\|x\| = \sqrt{\sum_{n=0}^{\infty} x_n^2}$ for any $x = (x_0, x_1, x_2, \cdots) \in \ell_2$.[12]

Let A be a compact subset in X which contains the Bochner integral of Ψ, $\int_L \Psi \, \mathrm{d}\eta$. Note that $\int_L \Psi \, \mathrm{d}\eta$ is contained in the set $\{\sum_{n=0}^{\infty} a_n x_n \colon |a_n| \leq 2^{-n}\}$, which is norm compact, hence such a set A can certainly be found. Let $M = \max\{\|x\| \colon x \in A\}$. It is clear that $M \geq \|e\| = 1$ since $e \in \int_L \Psi \, \mathrm{d}\eta \subseteq A$. Moreover, for any $b \in \overline{\mathrm{con}}(A)$, we have $\|b\| \leq M$ and $\|b - e/2\| \leq \|b\| + \|e/2\| < 2M$. Let $\beta = 1/(2M)$. Note also that $\beta \, \mathrm{d}(b, e/2) = \frac{1}{2M} \|b - e/2\| \leq 1$ for all $b \in \overline{\mathrm{con}}(A)$.

4.3.2 A counterexample

Recall that $\mathcal{U}_A$ denotes the space of continuous functions on $A \times \overline{\mathrm{con}}(A)$. Define a function $V \colon L \to \mathcal{U}_A$ as follows: for any $t \in L$, $a \in A$ and $b \in \overline{\mathrm{con}}(A)$,

$$V(t)(a, b) = -h(t, a, \psi(t), \beta \, \mathrm{d}(b, e/2)) - \|a\| \cdot \|a - \psi(t)\|, \tag{4.5}$$

where $h \colon L \times X \times X \times \mathbb{R}_+ \to \mathbb{R}_+$ is a function to be defined as below. For any $t \in L = [0, 1]$, $x, y \in X$ and $\ell \geq 0$,

$$h(t, x, y, \ell) = \begin{cases} \ell \left|\sin \frac{t}{\ell}\pi\right| \left(\|x\| + 1 - (-1)^{\left[\frac{t}{\ell}\right]}\right) \left(\|x - y\| + 1 + (-1)^{\left[\frac{t}{\ell}\right]}\right), & \text{if } \ell > 0; \\ 0, & \text{if } \ell = 0. \end{cases} \tag{4.6}$$

Applying the same argument as in the proof of Lemma 3 in Khan, Rath and Sun (1997), $V \colon L \to \mathcal{U}_A$ is a $\mathcal{L}$-measurable function.

Now we are ready to specify our nonatomic game $\mathcal{G}_1 \colon L \to \mathcal{U}_A$. The player space is modeled by the Lebesgue unit interval $(L, \mathcal{L}, \eta)$. The set $A \subseteq X$ above, which contains the integral of the correspondence Ψ, serves as the common action set for all the players in this game. Meanwhile, the closed convex hull of A, $\overline{\mathrm{con}}(A)$ serves as the set of societal responses. For any player $t \in L$, suppose that this player's own action is $a \in A$ and the

[12] For more discussion on the Lyapunov example on the range of vector measures, see *e.g.*, Khan and Zhang (2012a).

societal response is $b \in \overline{\text{con}}(A)$, player t's payoff is

$$\mathcal{G}_1(t)\,(a,b) = V(t)\,(a,b)\,, \tag{4.7}$$

where V is defined in Eq. (4.5).

The following is our main result in this section. The proof is given in Section 4.6.

Theorem 4.3.3. *There is no pure-strategy Nash equilibrium in the game $\mathcal{G}_1$.*

Remark 4.3.4. Here is the key idea in the construction of the nonatomic game $\mathcal{G}_1$ and the game in Section 4 of Khan, Rath and Sun (1997). Based on the particular payoff functions in Eq. (4.7), if there exists a pure-strategy Nash equilibrium in $\mathcal{G}_1$, then the induced societal response of this equilibrium must be $e/2$. Meanwhile, when facing this special societal response $e/2$, for η-almost all player t, the best response is either 0 or $\psi(t)$. In other words, if there exists a pure-strategy Nash equilibrium of the game $\mathcal{G}_1$, it must be a $\mathcal{L}$-measurable selection of the correspondence Ψ in Eq. (4.4) and its Bochner integral is $e/2$. However, such a selection does not exist according to Lemma 4.3.1. It is worthwhile to note that the idea which extends the counterexample for ℓ_2 to a counterexample for arbitrary infinite-dimensional Banach spaces has been roughly mentioned by Khan, Rath and Sun (1997).[13]

4.3.3 More examples

For any $s \in (0,1)$, we next construct a nonatomic game $\mathcal{G}_s\colon (L,\mathcal{L},\eta) \to \mathcal{U}_A$ such that there is no pure-strategy Nash equilibrium either. For any player $t \in L$, the payoff function of this player is defined to be a function $\mathcal{G}_s(t)\colon A \times \overline{\text{con}}(A) \to \mathbb{R}$ as follows, for every $a \in A$ and every societal response $b \in \overline{\text{con}}(A)$,

$$\mathcal{G}_s(t)(a,b) = \begin{cases} \mathcal{G}_1\left(\frac{t}{s}\right)\left(a,\frac{b}{s}\right), & \text{if } t \in [0,s] \text{ and } \frac{b}{s} \in \overline{\text{con}}(A); \\ \mathcal{G}_1\left(\frac{t}{s}\right)(a, c\cdot b)\,, & \text{if } t \in [0,s] \text{ and } \frac{b}{s} \notin \overline{\text{con}}(A); \\ -\|a\|, & \text{if } t \in (s,1]. \end{cases} \tag{4.8}$$

where $\mathcal{G}_1(\cdot)$ is the payoff function of the game $\mathcal{G}_1$ in Eq. (4.7), and $c \cdot b$ is the intersection between the boundary of $\overline{\text{con}}(A)$ and the ray from 0 to b. Notice that for any $t \in L$, $\mathcal{G}_s(t)$

[13] See p. 33 in Khan, Rath and Sun (1997), "by an appealing to Corollary 6 (p. 265) in Diestel and Uhl (1977), one should hopefully be able to set a version of the counterexample in any arbitrary infinite-dimensional Banach space."

is a continuous function on $A \times \overline{\text{con}}(A)$. And it is easy to check that $\mathcal{G}_s$ is a Lebesgue measurable function from the Lebesgue unit interval to $\mathcal{U}_A$.

Corollary 4.3.5. *For any $s \in (0, 1]$, there is no pure-strategy Nash equilibrium in $\mathcal{G}_s$.*

4.4 Saturation and games

If players take actions from an arbitrary infinite-dimensional Banach space, if the player space is modeled by the Lebesgue unit interval, the nonatomic games in Section 4.3 do not have pure-strategy Nash equilibria. In contrast, Khan and Sun (1999) find that if the player space is modeled by a nonatomic Loeb space (see Theorem 2 therein), there always exist pure-strategy Nash equilibria in such nonatomic games. This section answers the question that which property of the Lebesgue unit interval (or of nonatomic Loeb spaces) is responsible to the failure (or success) about the existence of the pure-strategy Nash equilibria. We find that the underlying player space being saturated is a sufficient and necessary condition for the existence of pure-strategy Nash equilibria in nonatomic games; the sufficiency and necessity results are presented in Sections 4.4.1 and 4.4.2 respectively.

4.4.1 The sufficiency result

In this section, we assume that $(X, \|\cdot\|)$ is an infinite-dimensional Banach space, and A is a nonempty compact subset of it. As before, let $\mathcal{U}_A$ be the space of continuous real-valued functions on $A \times \overline{\text{con}}(A)$ endowed with both the sup-norm topology and the corresponding Borel σ-algebra (generated by this topology).

Proposition 4.4.1. *If $(T, \mathcal{T}, \mu)$ is a saturated probability space and $\mathcal{G}$ is a $\mathcal{T}$-measurable map from T to $\mathcal{U}_A$, then the game $\mathcal{G}$ has a pure-strategy Nash equilibrium.*[14]

The following three remarks provide three straightforward but different proofs for Proposition 4.4.1 based on the existing results.

Remark 4.4.2. Proposition 4.4.1 can be proved by transferring the existing results on nonatomic Loeb spaces (see Theorem 2 in Khan and Sun (1999)) via the saturation

[14] A result similar to Proposition 4.4.1 is also obtained independently by Yu (2012), see Lemma 3 therein.

property defined in Remark 3.[15] For the game $\mathcal{G}$ on the saturated probability space $(T, \mathcal{T}, \mu)$, we have a game $\mathcal{F}$ on a nonatomic Loeb space such that $\mathcal{F}$ has the same distribution with $\mathcal{G}$. By Theorem 2 in Khan and Sun (1999), there is a pure-strategy Nash equilibrium f of the game $\mathcal{F}$. The saturation property implies the existence of $g\colon T \to A$, such that the joint distribution of $\mathcal{F}$ and f is same as the joint distribution of $\mathcal{G}$ and g. It is not difficult to see that g is a pure-strategy Nash equilibrium of the game $\mathcal{G}$.

Remark 4.4.3. This proposition can also be proved via the existing result on nonatomic games where the societal response is formulated as the distribution. Let $\hat{\mathcal{U}}_A$ be the space of real-valued continuous functions on $A \times \mathcal{M}(A)$ endowed with the sup-norm topology, where $\mathcal{M}(A)$ is the space of all Borel probability measures on A with Prohorov metric. A new game $\hat{\mathcal{G}}\colon (T, \mathcal{T}, \mu) \to \hat{\mathcal{U}}_A$ can be defined as follows: for each player t, $a \in A$ and $\tau \in \mathcal{M}(A)$,

$$\hat{\mathcal{G}}(t)(a, \tau) = \mathcal{G}(t)\left(a, \int_A \mathrm{Id}_A \, \mathrm{d}\tau\right),$$

where Id_A is the identity map on A. The existence of pure-strategy Nash equilibria in this model of nonatomic games is a trivial consequence of the saturation property from the earlier results on distributional equilibria or large games with a Loeb space of players; see the sufficiency part of Theorem 4.6 and Proposition 5.3(1) in Keisler and Sun (2009), and also Corollary 4(4) in Carmona and Podczeck (2009) and Theorem 2 of Noguchi (2009). Let $f\colon T \to A$ be a pure-strategy Nash equilibrium of $\hat{\mathcal{G}}$. This f is also a pure-strategy Nash equilibrium of the original game $\mathcal{G}$ because $\int_A \mathrm{Id}_A \, \mathrm{d}(\mu \circ f^{-1}) = \int_T f \, \mathrm{d}\mu$, where the equation holds according to the substitution of variables.

Remark 4.4.4. One can also show Proposition 4.4.1 by applying the fixed-point theorem of Fan (1952) and Glicksberg (1952). Towards this end, one needs to check the convexity, compactness and preservation of upper hemi-continuity for the integration of Banach space-valued correspondences on general saturated probability spaces.

We now briefly review the existing results on the integration theory of the Banach-valued correspondences. Khan and Majumdar (1986) and Yannelis (1990) consider the approximate versions of Fatou's lemma. However, one should note that Proposition 4.4.1 needs the exact version of Fatou's lemma which guarantees the preservation of upper hemi-continuity. Rustichini and Yannelis (1991) show the properties of convexity, compactness and preservation of upper hemi-continuity[16] for the Bochner integral of a cor-

[15] For more discussion on this "transferring" technique, see Section 5 of Keisler and Sun (2009).

[16] Based on the compactness result, Yannelis (1990) points out "preservation of upper hemi-continuity"

respondence from a probability space $(T, \mathcal{T}, \mu)$ to a fixed weakly compact set in an infinite-dimensional Banach space, under the condition that for each nonnegligible set $E \in \mathcal{T}$, the cardinality of $L_E^\infty(\mu)$ is larger than the continuum. Sun (1997) proves those properties for the integration of general correspondences on nonatomic Loeb spaces. Furthermore, Sun and Yannelis (2008b) find that all the existing results on Loeb spaces can be transferred to results on saturated spaces via the saturation property easily. Moreover, Podczeck (2008) shows the convexity and compactness results over general saturated probability spaces without appealing to the existing relevant results on Loeb spaces.

Remark 4.4.5. Other different models of nonatomic game with actions in an infinite-dimensional spaces are also considered in Khan, Rath and Sun (1997) and Khan and Sun (1999). First, instead of taking actions from a norm-compact subset, the players in the nonatomic game in Theorem 2 of Khan and Sun (1999) can take actions from a weakly compact subset of a separable Banach space and this nonatomic game is a weakly continuous function. Note that a norm-compact set is also a weakly compact set in a Banach space and norm-continuous functions on norm-compact sets are weakly continuous. As a result, the model in Khan and Sun (1999) is more general than the model in this paper. Nevertheless, our Proposition 4.4.1 above still holds for this more general model, provided that the Banach space is separable. [17] It is in this sense that we claim that our Proposition 4.4.1 generalizes Theorem 2 of Khan and Sun (1999).

Second, Section 6 of Khan, Rath and Sun (1997) also consider a setting of nonatomic games where players taking actions from a weak* compact subset in the dual space of a separable Banach space. In this setting, the societal response is formulated as the Gel'fand integral of the strategy profile and a nonatomic game is a weak* continuous function. This modeling is closely related to the nonatomic games with actions in compact metric spaces, and the societal response is formulated as the action distribution of the strategy profile; see Khan, Rath and Sun (1997) for more discussion. The existence of pure-strategy Nash equilibria in this model can be obtained by applying the approach in

for the Bochner integral of correspondences can be proved directly; see Theorem 3.1 and Remark 3.1 therein.

[17]This claim follows from the Fan-Glicksberg's fixed-point argument and the results in Sun and Yannelis (2008b) on the integration of weakly compact Banach-valued correspondences over the saturated spaces as discussed in Remark 4.4.3; the corresponding convexity and the compactness results are also proved by Podczeck (2008). It is worthwhile to note that the separability is necessary for the preservation of upper hemi-continuity property. This claim also follows directly from the existing results for the games with the action distribution as the societal response (see Remark 4.4.4), and separability guarantees the metrizability of weak compact action subset.

Remark 4.4.3 or the standard Fan-Glicksberg fixed-point argument as in Remark 4.4.4. It is worthwhile to note that the separability of the Banach spaces is required, either to guarantee the metrizability of the action space when applying Remark 4.4.3, or to obtain the preservation of the upper hemi-continuity property when using the fixed-point arguments.

Remark 4.4.6. In Yu and Zhu (2005), a nonatomic game model is presented where the players' payoffs depend on their own actions and the average of certain transformation of the strategy profile for all the other players, and the transformation is a continuous map transforming actions into some statistics data in a finite-dimensional space. According to Proposition 4.4.1, the results in Yu and Zhu (2005) can be generalized to infinite-dimensional Banach spaces, provided that the player space is modeled by a saturated probability space.

4.4.2 The necessity result

In this section, $(X, \|\cdot\|)$ is any infinite-dimensional Banach space, and A is fixed to be the compact subset of X which contains the integral of the correspondence $\Psi\colon L \twoheadrightarrow X$ as in Eq. (4.4). We now present the following result.

Theorem 4.4.7. *Fix a nonatomic probability space $(T, \mathcal{T}, \mu)$ and the compact set A. If there exists a pure-strategy Nash equilibrium for any nonatomic game with the player space $(T, \mathcal{T}, \mu)$ and the common action set A, then $(T, \mathcal{T}, \mu)$ is a saturated probability space.*

Here is a claim equivalent to this Theorem. If the player space is modeled by a nonsaturated probability space, then there is a nonatomic game based on this player space such that there does not exist a pure-strategy Nash equilibrium. The following is a sketch for the construction of the nonatomic game in this situation.

First, note that the player space is modeled by a non-saturated probability space $(T, \mathcal{T}, \mu)$, by Definition 2.2.1, there is a subset S with measure $s = \mu(S) > 0$, such that the probability space restricted to S is countably-generated modulo the null sets. As a result of Maharam's theorem (see Maharam (1942)), the measure algebra of the restricted measure space over S is isomorphic to that of measure algebra of the Lebesgue subinterval over $[0, s]$. It is a well-known result that there is a measure-preserving map q

from the measure space restricted to S to the Lebesgue interval $[0, s]$ such that q induces the above measure-algebra isomorphism; see Theorem 4.12 (p. 937) of Fremlin (1989).

Next, a nonatomic game can be constructed by the composition of the nonatomic game $\mathcal{G}_s$ in Section 4.3 and the above mapping q. Denote the new nonatomic game by $\mathcal{G}'_s$, here the player space is $(T, \mathcal{T}, \mu)$. Note that the game $\mathcal{G}_s$ is defined on the Lebesgue unit interval and there is no pure-strategy Nash equilibrium; see Corollary 4.3.5. The construction of the mapping q can ensure us to transfer the nonexistence result from the game $\mathcal{G}_s$ to the new nonatomic game $\mathcal{G}'_s$. Suppose not, there is a pure-strategy Nash equilibrium in $\mathcal{G}'_s$, then one can construct a pure-strategy Nash equilibrium in $\mathcal{G}_s$. This contradicts the statement in Corollary 4.3.5.

Remark 4.4.8. In Theorem 4.4.7, we show that the underlying player space being a saturated probability space serves a necessity condition for the existence of pure-strategy Nash equilibrium for nonatomic games with actions in infinite-dimensional Banach spaces. A similar necessity result is presented in Theorem 4.6 of Keisler and Sun (2009) in the context of nonatomic games, where the common action set is an uncountable compact metric space and the societal response is the action distribution. Though Proposition 4.4.1 is a straightforward result of the sufficiency part of Theorem 4.6 of Keisler and Sun (2009) as in Remark 4.4.3, Theorem 4.4.7 is not implied by the necessity part of Theorem 4.6 of Keisler and Sun (2009).

4.5 Discussion

In this paper, we find that the player space being a saturated probability space is a "sufficient and necessary" condition for the existence of pure-strategy Nash equilibria in nonatomic games with actions in infinite-dimensional Banach spaces.[18] This property is sufficient in the sense that if the player space is a saturated probability space, then every such nonatomic game has a pure-strategy Nash equilibrium, and it is necessary that if every nonatomic game has a pure-strategy Nash equilibrium, then the player space must be a saturated probability space. This finding answers why one can not establish a general result on the existence of pure-strategy Nash equilibria in nonatomic games on the Lebesgue unit interval as in Theorem 4.3.3 but one can make it on nonatomic Loeb

[18] In Khan and Zhang (2012b), "sufficient and necessary" results on the property of saturation are established in the context of finite-player games with diffused private information.

spaces as in Khan and Sun (1999). It is simply because that the Lebesgue unit interval is not saturated but nonatomic Loeb spaces are.

However, as far as the counterexamples $\mathcal{G}_s, s \in (0,1]$ in Section 4.3 are concerned, to guarantee the existence of pure-strategy Nash equilibria in such nonatomic games, it is not necessary to turn to model the player space by a saturated probability space, while a simpler essentially countably-generated probability space does work. The Lebesgue extension in Khan and Zhang (2012a) is a candidate to serve this purpose.

We first briefly review the construction of this Lebesgue extension. Let $K = [0,1]$, and $(K, \mathcal{K}, \kappa)$ another copy of the Lebesgue interval $(L = [0,1], \mathcal{L}, \eta)$. First, there is partition of L, denoted by $\{C_k \subseteq [0,1] \colon k \in K\}$, such that $\eta_*(C_k) = 0$ and $\eta^*(C_k) = 1$ for each k where η_* and η^* are the corresponding inner and outer measure associated to the Lebesgue measure η. Let $C = \{(i,k) \in L \times K \colon i \in C_k, k \in K\}$. It is clear that $C \subseteq L \times K$, and C has $\eta \otimes \kappa$-inner measure 0 and outer measure 1. Second, the Lebesgue σ-algebra $\mathcal{L} \otimes \mathcal{K}$ can be extended which contains C and C has measure 1 in this extension. There is no harm to restrict this extension to C to get a probability structure on C. Finally, note that the projection from C to L is a one-to-one mapping, a probability structure can be obtained on L through this projection and the probability structure on C. It turns out that the resulted probability structure on L is an essentially countably-generated Lebesgue extension. Denote this countably-generated Lebesgue extension by $(I, \mathcal{I}, \lambda)$.

Proposition 4.5.1. *If the player space is modeled by the countably-generated extended Lebesgue interval $(I, \mathcal{I}, \lambda)$, there exists a pure-strategy Nash equilibrium in $\mathcal{G}_s$ for all $s \in (0,1]$.*

The proof of this positive result is moved to Section 4.6.

Remark 4.5.2. It is worthwhile to note that we do not appeal to the results in Proposition 4.4.1 to prove Proposition 4.5.1. For a fixed nonatomic game $\mathcal{G}_s$, if the player space is modeled by a saturated probability space $([0,1], \mathcal{T}, \mu)$, it follows from Proposition 4.4.1 that there does exist a pure-strategy Nash equilibrium, say $f_s \colon [0,1] \to A$, which is $\mathcal{T}$-measurable but not Lebesgue-measurable. Let $\mathcal{I}_s$ be the smallest σ-algebra including both the Lebesgue σ-algebra $\mathcal{L}$ and the σ-algebra generated by f_s, which is an essentially countably-generated sub-σ-algebra of $\mathcal{T}$. It is clear that $\mathcal{I}_s$ is also essentially countably-generated and f_s is $\mathcal{I}_s$-measurable. However, for any $s \in (0,1]$, given any arbitrary f_s in the above sense, the smallest σ-algebra including both the Lebesgue

σ-algebra $\mathcal{L}$ and the σ-algebras generated by these f_s for all $s \in (0,1]$ could not be essentially countably generated.

4.6 Proofs

4.6.1 Proofs of results in Section 4.3

Proof of Lemma 4.3.1. We prove this result by contradiction. Suppose that $e/2 \in \int_L \Psi(t)\,\mathrm{d}\eta(t)$, that is, there is a selection $f\colon L \to X$ of Ψ such that $\int_L f\,\mathrm{d}\eta = e/2$. Here f is a $\mathcal{L}$-measurable function and for any $t \in L$, $f(t)$ is either 0 or $\psi(t)$. Let $E = \{t \in L\colon f(t) \neq 0\}$. It is clear that $E \in \mathcal{L}$ and $\int_L f\,\mathrm{d}\eta = \int_E \psi\,\mathrm{d}\eta$. As a result $e/2 = \int_E \psi(t)\,\mathrm{d}\eta(t)$, that is,

$$\frac{1}{2}\frac{x_0}{\|x_0\|} = \sum_{n=0}^{\infty} \frac{x_n}{2^n\|x_n\|}\int_E W_n(t)\,\mathrm{d}\eta(t). \tag{4.9}$$

Apply x_0^* on the both sides of Eq. (4.9), we will have $\eta(E) = 1/2$. Then for each integer $n \geq 1$, apply x_n^* on the both sides of Eq. (4.9), we will have

$$\eta(E \cap E_n) = \eta(E \cap E_n^c),$$

where E_n^c is the complement of E_n.

Let $g = \chi_E - \chi_{E^c}$, where χ_E is the indicator function of E. Then $\int_L W_n(t)g(t)\,\mathrm{d}\eta(t) = 0$ for each integer $n \geq 0$. Because that the Walsh system is a complete orthogonal basis of the space of all square integrable functions on the Lebesgue unit interval, $g = 0$, hence it is a contradiction to the definition of g. Therefore $e/2 \notin \int_L \Psi\,\mathrm{d}\eta$. □

Before offering a proof of Theorem 4.3.3, we shall need three additional inequalities.

Lemma 4.6.1. *For $\alpha > 0$ and any real numbers b_1, b_2 and c, with $b_1 < b_2$, we have*

$$\left|\int_{b_1}^{b_2} (-1)^{\left[\frac{t+c}{\ell}\right]}\,\mathrm{d}\eta(t)\right| \leq \ell.$$

Proof. Routine. □

Lemma 4.6.2. *For $\ell > 0$, and any integer $n \geq 0$,*

$$\left| \int_0^1 (-1)^{[\frac{t}{\ell}]} W_n(t)\, \mathrm{d}\eta(t) \right| \leq \min\{1, 2n\ell\}.$$

Proof. It is obvious that

$$\left| \int_0^1 (-1)^{[\frac{t}{\ell}]} W_n(t)\, \mathrm{d}\eta(t) \right| \leq 1.$$

We next show the other part. Consider the binary representation of n,

$$n = n_0 + 2n_1 + 2^2 n_2 + \cdots + 2^{m-1} n_{m-1},$$

where $n_{m-1} = 1$ and $n_0, n_1, \ldots, n_{m-2}$ are either 0 or 1. For any $t \in \left(\frac{s-1}{2^m}, \frac{s}{2^m}\right)$, where s is an integer from $\{1, 2, \ldots, 2^m\}$, it is clear that the first m numbers in the binary representation of t are fixed. By definition, $W_n(t) = (-1)^{c_s}$ over $\left(\frac{s-1}{2^m}, \frac{s}{2^m}\right)$, where c_s is a constant integer. Then we have

$$\left| \int_0^1 (-1)^{[\frac{t}{\ell}]} W_n(t)\, \mathrm{d}\eta(t) \right| \leq \sum_{s=1}^{2^m} \left| \int_{\frac{s-1}{2^m}}^{\frac{s}{2^m}} (-1)^{[\frac{t}{\ell}]} W_n(t)\, \mathrm{d}\eta(t) \right| = \sum_{s=1}^{2^m} \left| \int_{\frac{s-1}{2^m}}^{\frac{s}{2^m}} (-1)^{[\frac{t+c_s\ell}{\ell}]}\, \mathrm{d}\eta(t) \right|.$$

It follows from Lemma 4.6.1 that

$$\left| \int_0^1 (-1)^{[\frac{t}{\ell}]} W_n(t)\, \mathrm{d}\eta(t) \right| \leq \sum_{s=1}^{2^m} \ell = 2^m \ell \leq 2n\ell,$$

where the last equality holds because of the binary representation of n. □

Lemma 4.6.3. *For $\ell > 0$,*

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \left| \int_0^1 (-1)^{[\frac{t}{\ell}]} W_n(t)\, \mathrm{d}\eta(t) \right| < 2\ell.$$

Proof. Take $r = [1/(2\ell)]$, then $r \leq 1/\ell$ and $r + 1 > 1/(2\ell)$. By Lemma 4.6.2, we will have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \left| \int_0^1 (-1)^{\left[\frac{t}{\ell}\right]} W_n(t)\,\mathrm{d}\eta(t) \right| &\le \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \min\{1, 2n\ell\} \\
&\le \sum_{n=0}^{r} \frac{1}{2^{n+1}} 2n\ell + \sum_{n=r+1}^{\infty} \frac{1}{2^{n+1}} \\
&= \ell \left(2 - \frac{r+2}{2^r}\right) + \frac{1}{2^{r+1}} \\
&= 2\ell - \frac{2(r+2)\ell - 1}{2^{r+1}} < 2\ell.
\end{aligned}$$

□

Proof of Theorem 4.3.3. We will prove this result by contradiction. Suppose that there exists a pure-strategy Nash equilibrium in the game $\mathcal{G}_1$, and assume that the $\mathcal{L}$-measurable function $f\colon L \to A$ is a pure-strategy Nash equilibrium. We next prove that the following two cases can not happen, (1) $\int_L f(t)\,\mathrm{d}\eta(t) = e/2$ and (2) $\int_L f(t)\,\mathrm{d}\eta(t) \neq e/2$, where $\int_L f(t)\,\mathrm{d}\eta(t)$ is the societal response when players take actions as in the Nash equilibrium f and $e = x_0/\|x_0\|$ as in Eq. (4.3). Therefore we obtain a contradiction. Next we discuss these two cases separately by dividing our arguments into the following two parts.

Part 1. Suppose that $\int_L f(t)\,\mathrm{d}\eta(t) = e/2$. Then for any player $t \in L$, the payoff function of player t reduces to the following form, for any $a \in A$,

$$u_t\left(a, \int_L f\,\mathrm{d}\eta\right) = u_t\left(a, e/2\right) = -\|a\| \cdot \|a - \psi(t)\|, \tag{4.10}$$

As a result, the best response of player t is the doubleton set $\{0, \psi(t)\}$ for any $t \in L$. Hence the pure-strategy Nash equilibrium f, as a function from the Lebesgue unit interval to A, is a measurable selection of the correspondence Ψ in Eq. (4.4). However, Lemma 4.3.1 ensures that there does not exist a Lebesgue measurable selection from this correspondence Ψ whose Bochner integral is $e/2$. Hence it is a contradiction that $\int_L f(t)\,\mathrm{d}\eta(t) = e/2$.

Part 2. Suppose that $\int_L f(t)\,\mathrm{d}\eta(t) \neq e/2$. That is, in the pure-strategy Nash equilibrium $f\colon L \to A$, the societal response is no longer $e/2$. As a result, in the first item

h, see Eq. (4.6), of the payoff function for any player $t \in L$, the fourth argument is no longer zero. Let

$$\ell_0 = \beta d\left(\int_L f(t)\,\mathrm{d}\eta(t), \frac{e}{2}\right).$$

Certainly, $0 < \ell_0 \leq 1$. Divide the unit interval L into intervals of length ℓ_0, so that we obtain $L \cap \big(\cup_{n\in\mathbb{N}} [n\ell_0, (n+1)\ell_0)\big)$, where $\mathbb{N}$ is the set of all nonnegative integers.

We first characterize the set of best responses for any player when other players take actions in the Nash equilibrium f, *i.e.*, the societal response is $\int_L f\,\mathrm{d}\eta$. On the one hand, we first fix a player $t \in (n\ell_0, (n+1)\ell_0)$. If n is even, then the integer part of t/ℓ_0, $[t/\ell_0]$, is even, as a result the payoff function of this player t (see Eq. (4.7)) reduces to the following form, for any $a \in A$,

$$u_t\left(a, \int_L f\,\mathrm{d}\eta\right) = -\ell_0\left|\sin\frac{t}{\ell_0}\pi\right| \cdot \|a\| \cdot (\|a-\psi(t)\| + 2) - \|a\| \cdot \|a-\psi(t)\|.$$

Note that the value of $u_t(a, \int_L f\,\mathrm{d}\eta) \leq 0$ for any $a \in A$, and the value is 0 if and only if $a = 0$. As a result, $u_t(a, \int_L f\,\mathrm{d}\eta)$ takes the maximum value only at $a = 0$. That is, the best response for this player t, when facing the societal response $\int_L f\,\mathrm{d}\eta$, is the singleton set $\{0\}$. Similarly, for player $t \in (n\ell_0, (n+1)\ell_0)$, if n is an odd natural number, so is the integer part $[t/\ell_0]$. As a result, the payoff function of this player t reduces to the following form, for any $a \in A$,

$$u_t\left(a, \int_L f\,\mathrm{d}\eta\right) = -\ell_0\left|\sin\frac{t}{\ell_0}\pi\right| \cdot (\|a\| + 2) \cdot \|a-\psi(t)\| - \|a\| \cdot \|a-\psi(t)\|.$$

Using the similar argument as above as n is even, we can obtain that the best response for this player t, when facing the societal response $\int_L f\,\mathrm{d}\eta$, is the singleton set $\{\psi(t)\}$.

On the other hand, we consider the player $t = n\ell_0$ for some $n \in \mathbb{N}$. In this case, the payoff function of player t (see Eq. (4.7)) reduces to the following form, for any $a \in A$,

$$u_t\left(a, \int_L f\,\mathrm{d}\eta\right) = -\|a\| \cdot \|a-\psi(t)\|.$$

It is clear that the set of best responses for this player t, when facing the societal response $\int_L f\,\mathrm{d}\eta$, is a doubleton set $\{0, \psi(t)\}$.

To summarize, except for the players in the set $\{t = n\ell_0 \in L \colon n \in \mathbb{N}\}$, the best response for any other player is a singleton set. Note also that $\{t = n\ell_0 \colon n \in \mathbb{N}\}$ is a η-null set in the Lebesgue interval. As a result, the pure-strategy Nash equilibrium $f\colon L \to A$ is of the following form, for almost all $t \in L$,

$$f(t) = \frac{1 - (-1)^{\left[\frac{t}{\ell_0}\right]}}{2}\psi(t). \tag{4.11}$$

Next we calculate the distance between the societal response $\int_L f\,\mathrm{d}\eta$ and $e/2$, where the pure-strategy Nash equilibrium f is determined in Eq. (4.11) above.

$$\begin{aligned}
d\left(\int_L f(t)\,\mathrm{d}\eta(t), \frac{e}{2}\right) &= \left\| \int_L f(t)\,\mathrm{d}\eta(t) - \frac{e}{2} \right\| \\
&= \left\| \int_L f(t)\,\mathrm{d}\eta(t) - \frac{1}{2}\int_0^1 \psi(t)\,\mathrm{d}\eta(t) \right\| \\
&= \left\| \frac{1}{2}\int_L (-1)^{\left[\frac{t}{\ell_0}\right]}\psi(t)\,\mathrm{d}\eta(t) \right\| \\
&= \left\| \frac{1}{2}\int_L (-1)^{\left[\frac{t}{\ell_0}\right]} \sum_{n=0}^{\infty} \frac{x_n}{2^n\|x_n\|} W_n(t)\,\mathrm{d}\eta(t) \right\| \\
&= \left\| \sum_{n=0}^{\infty} \int_L \frac{1}{2}(-1)^{\left[\frac{t}{\ell_0}\right]} \frac{x_n}{2^n\|x_n\|} W_n(t)\,\mathrm{d}\eta(t) \right\| \\
&\leq \sum_{n=0}^{\infty} \frac{1}{2} \left\| \int_L (-1)^{\left[\frac{t}{\ell_0}\right]} \frac{x_n}{2^n\|x_n\|} W_n(t)\,\mathrm{d}\eta(t) \right\| \\
&\leq \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \left| \int_L (-1)^{\left[\frac{t}{\ell_0}\right]} W_n(t)\,\mathrm{d}\eta(t) \right|
\end{aligned}$$

By virtue of the inequality in Lemma 4.6.3, we have

$$d\left(\int_L f(t)\,\mathrm{d}\eta(t), \frac{e}{2}\right) < \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} 2\ell_0 = 2\ell_0. \tag{4.12}$$

Notice that $\ell_0 = \beta\,\mathrm{d}(\int f, e/2)$, where $\beta = 1/(2M)$ and $M = \max\{\|a\|\colon a \in A\}$. Note that $M \geq 1$, it follows that $\beta \leq 1/2$. It follows from Eq. (4.12) that $\mathrm{d}(\int f\,\mathrm{d}\eta, e/2) < 2\ell_0 = 2\beta\,\mathrm{d}(\int f\,\mathrm{d}\eta, e/2)$. This implies that $\beta > 1/2$. It is a contradiction with $\beta \leq 1/2$. Therefore we complete the proof of Part 2 that there does not exist a pure-strategy Nash equilibrium $f\colon L \to A$ with $\int_L f\,\mathrm{d}\eta \neq e/2$. □

Proof of Corollary 4.3.5. We show it by contradiction. Suppose that $f\colon [0,1] \to A$ is a pure-strategy Nash equilibrium in the game $\mathcal{G}_s$. Note that for any player $t \in (s, 1]$, the best response is 0. As a result, in this Nash equilibrium f, the actions for players in $(s,1]$ do not affect the societal response. In particular,

$$\frac{1}{s}\int_L f \,\mathrm{d}\eta = \frac{1}{s}\int_0^s f \,\mathrm{d}\eta = \int_0^s f \,\mathrm{d}\eta^{[0,s]},$$

which is exactly an element in $\overline{\mathrm{con}}(A)$. Moreover, for any player $t \in [0,s]$, if the other players follow actions as in this Nash equilibrium f, for any $a \in A$,

$$\mathcal{G}_s(t)\left(a, \int_L f \,\mathrm{d}\eta\right) = \mathcal{G}_1\left(\frac{t}{s}\right)\left(a, \frac{1}{s}\int_0^s f \,\mathrm{d}\eta\right).$$

For η-almost all player $t \in [0,s]$, since f is a pure-strategy Nash equilibrium, $f(t)$ is a best response for this player t given others players follow f. That is, for any $a \in A$

$$\mathcal{G}_1\left(\frac{t}{s}\right)\left(f(t), \frac{1}{s}\int_0^s f \,\mathrm{d}\eta\right) \geq \mathcal{G}_1\left(\frac{t}{s}\right)\left(a, \frac{1}{s}\int_0^s f \,\mathrm{d}\eta\right). \tag{4.13}$$

Let $g\colon [0,1] \to A$ defined by $g(t') = f(t' \cdot s)$ for η-almost all $t' \in [0,1]$. It follows from substitution of variables that $\frac{1}{s}\int_0^s f(t)\,\mathrm{d}\eta(t) = \int_0^1 g(t')\,\mathrm{d}\eta(t')$. According to Eq. (4.13), for all $t' \in [0,1]$ and $a \in A$

$$\mathcal{G}_1(t')\left(g(t'), \int_0^1 g \,\mathrm{d}\eta\right) \geq \mathcal{G}_1(t')\left(a, \int_0^1 g \,\mathrm{d}\eta\right).$$

Hence g is a pure-strategy Nash equilibrium for the nonatomic game $\mathcal{G}_1$. It is a contradiction with Theorem 4.3.3. □

4.6.2 Proof of Theorem 4.4.7

We prove this theorem by contradiction. Suppose that $(T, \mathcal{T}, \mu)$ is not a saturated probability space, we will construct a game with the player space $(T, \mathcal{T}, \mu)$ and A the set of common actions such that there is no pure-strategy Nash equilibrium in this game.

Since $(T, \mathcal{T}, \mu)$ is not a saturated probability space, by Definition 2.2.1, there exists a nonnegligible subset $S \in \mathcal{T}$ with μ-measure $s \in (0,1]$ such that the restricted probability space $(S, \mathcal{T}^S, \mu^S)$ is countably generated modulo the null sets. As a result of Maharam's theorem (see Maharam, 1942), the measure algebra of the Lebesgue subinterval, $([0,s], \mathcal{L}^{[0,s]}, \eta^{[0,s]})$ is isomorphic to that of $(S, \mathcal{T}^S, \mu^S)$. It is a known result that this isomorphism can be realized by a measure-preserving map q from $(S, \mathcal{T}^S, \mu^S)$ to $([0,s], \mathcal{L}^{[0,s]}, \eta^{[0,s]})$; see Theorem 4.12 in Fremlin (1989, p. 937).

We are now ready to construct a game with the player space $(T, \mathcal{T}, \mu)$ such that there is no pure-strategy Nash equilibrium, denote it by $\mathcal{G}'_s$. For any player $t \in T$, suppose that the action of this player is $a \in A$ and the other players take actions following the action profile $g\colon (T, \mathcal{T}, \mu) \to A$, the payoff function of player t is defined as follows,

$$\mathcal{G}'_s(t)\left(a, \int_T g \,\mathrm{d}\mu\right) = \begin{cases} \mathcal{G}_s(q(t))\left(a, \int_T g \,\mathrm{d}\mu\right), & \text{if } t \in S; \\ -\|a\|, & \text{if } t \notin S. \end{cases}$$

where $\mathcal{G}_s$ is the game defined on the Lebesgue unit interval in Corollary 4.3.5; see Eq. (4.8).

Suppose that $g\colon T \to A$ is a pure-strategy Nash equilibrium for the game $\mathcal{G}'_s$. According to its payoff structure, it is clear that for any player $t \notin S$, the best response is 0. As a result, such players do not affect the societal response. In particular,

$$\frac{1}{s}\int_T g \,\mathrm{d}\mu = \frac{1}{s}\int_S g \,\mathrm{d}\mu = \int_S g \,\mathrm{d}\mu^S \in \overline{\mathrm{con}}(A).$$

As a result, for every player $t \in S$, the payoff function is, for all $a \in A$,

$$\mathcal{G}'_s(t)\left(a, \int_T g \,\mathrm{d}\mu\right) = \mathcal{G}_s(q(t))\left(a, \int_T g \,\mathrm{d}\mu\right) = \mathcal{G}_1\left(\frac{q(t)}{s}\right)\left(a, \int_S g \,\mathrm{d}\mu^S\right),$$

where the second equation follows from Eq. (4.8).

Since g is a pure-strategy Nash equilibrium in the game $\mathcal{G}'_s$, it follows from the proof of Theorem 4.3.3 (see, *e.g.*, Eqs. (4.10) and (4.11)) that for μ-almost all player $t \in S$, the best response $g(t)$ is either $\psi\left(q(t)/s\right)$ or 0. Let S_1 denote the set of players in S whose best response is nonzero, *i.e.*, $S_1 = \{t \in S\colon g(t) \neq 0\}$. Since g is $\mathcal{T}$-measurable, $S_1 \in \mathcal{T}$. Note that q induces a measure-algebra isomorphism over subspaces restricted to S and $[0,s]$. By the definition of q, there exists a Lebesgue subset $E \in [0,s]$ such that $q^{-1}(E) \in \mathcal{T}$ and S_1 differ up to a μ-null set, *i.e.*, $\mu[S_1 \Delta q^{-1}(E)] = 0$ where Δ is the symmetric difference operator in $\mathcal{T}$.

Now define $g' \colon [0,1] \to A$ as follows,

$$g'(t') = \begin{cases} \psi\left(\frac{t'}{s}\right), & \text{if } t' \in E; \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that g' is a $\mathcal{L}$-measurable map and $g'(q(t)) = g(t)$ for μ-almost all $t \in S$. Moreover,

$$\int_T g \, \mathrm{d}\mu = \int_{S_1} g \, \mathrm{d}\mu = \int_{q^{-1}(E)} g'(q) \, \mathrm{d}\mu = \int_E g' \, \mathrm{d}\eta = \int_0^1 g' \, \mathrm{d}\eta,$$

where the second equation follows from $\mu\left(S_1 \Delta q^{-1}(E)\right) = 0$, the third from substitution of variables and $\mu q^{-1} = \eta$, and the first and last from the fact that g and g' are 0 almost elsewhere.

It is straightforward to check that $g' \colon [0,1] \to A$ is a pure-strategy Nash equilibrium of the nonatomic game $\mathcal{G}_s$. A contradiction to Corollary 4.3.5.

4.6.3 Proof of Proposition 4.5.1

It is sufficient to prove the result for the game $\mathcal{G}_1$. By Lemma 2 of Khan and Zhang (2012a), there exists a subset $S \in \mathcal{I}$ such that for any $t \in [0,1]$, $\lambda([0,t] \cap S) = t/2$. Recall that $E_0 = [0,1]$ and for $n \geq 1$, E_n is a union of some subintervals of the Lebesgue unit interval with $\eta(E_n) = 1/2$. As a result, $\lambda(E_n \cap S) = \lambda(E_n)/2$ for all $n \in \mathbb{N}$. Since that $(I, \mathcal{I}, \lambda)$ is the extended Lebesgue interval, $E_n \in \mathcal{I}$ and $\lambda(E_n) = \eta(E_n)$ for all n. Now define a function $f \colon (I, \mathcal{I}, \lambda) \to X$ as follows,

$$f(t) = \begin{cases} \psi(t), & \text{if } t \in S; \\ 0, & \text{if } t \notin S. \end{cases}$$

Note that f is an $\mathcal{I}$-measurable function since $\psi \colon I \to X$ is a $\mathcal{L}$-measurable function and $(I, \mathcal{I}, \lambda)$ is an extended Lebesgue interval. It is easy to calculate that $\int_I f \, \mathrm{d}\lambda = e/2$. It is routine to prove that the $\mathcal{I}$-measurable function f is a pure-strategy Nash equilibrium for the game $\mathcal{G}_1$ if the player space is upgraded by the extended Lebesgue interval $(I, \mathcal{I}, \lambda)$.

Chapter 5

Equilibrium, core and insurance equilibrium in a private information economy

5.1 Introduction

Economic decisions are essentially made based on a decision maker's vision of the future. The future is not known yet, hence all decisions are made with some degree of uncertainty. However, these decisions are not made entirely blindfolded. Agents rely on available information in plotting future plan, and information is asymmetric to agents. The classical Arrow-Debreu-McKenzie model has been extended to reflect these two facts, namely, uncertainty and informational asymmetry.

The first attempt to introduce uncertainty in Arrow-Debreu-McKenzie model was made by Arrow (1964) and Debreu (1959), who introduced a state-contingent claims model in which agents' utility function and initial endowment are contingent on the underlying state of nature. By treating a same commodity in two states of nature as different types of commodities, their model can be naturally mapped to a deterministic economy model to which standard techniques and results apply.

Radner (1968) further extended Arrow-Debreu's model to allow for asymmetric information. In Radner's model, each agent possesses a piece of private information which partially reveals the true state of nature. While Radner's model has the feature of uncertainty and informational asymmetry, no genuine perfect competition exists for each individual agent has non-negligible influence in such a finite-agent model.

Based on Radner's private information economy model and Aumann's large deterministic economy model (see Aumann (1964)), Sun and Yannelis (see Sun (2006), Sun and Yannelis (2007a), Sun and Yannelis (2008a)) introduced a private information economy model with a continuum of agents. In the model, agents have no direct knowledge of the underlying uncertainty. Instead, they are informed of a noisy private information signal giving them a clue about the real state of nature. Informational negligibility prevails in their model.

For the private information economy model, various solution concepts have been put forward that parallel the standard notions in a deterministic economy model. Radner (1968) introduced Radner equilibrium (a.k.a. Walrasian expectations equilibrium). In a Radner equilibrium, commodity prices vary over the states of nature. Each agent makes a state contingent consumption plan to maximize her expected utility, subject to her interim budget set. While Radner's notion of equilibrium has been unanimously accepted in the literature as the extension of the classic Walrasian equilibrium for the private information economy model, the situation is more complicated with the notion of core.

The complication is mainly due to the fact that in a private information economy, members of a coalition may exchange information for their good. Several definitions of core for the private information economy model have thus been proposed depending on the amount of information to be shared in a coalition. Wilson (1978) (see also Kobayashi (1980)) introduced the notion of coarse core with a minimal use of information that is common to all coalition members. Yannelis (1991) formulated the concept of private core in which each agent uses, and is limited to, her/his own private information.

Another notion of equilibrium that also deserves some attention is the so-called insurance equilibrium. This equilibrium is used to study insurance systems where each agent takes on individual risks and makes choices of consumption to spread risks across states of nature. In the insurance equilibrium model, agents can transfer income from one state to another through insurance against mishaps in the future. Therefore, in the model, an agent's budget set is not limited to the income in each state. This model was studied in the large finite-agent setting by Malinvaud (1972) and the continuum agent setting by Sun (2006). The latter paper further investigated the issue of insurability in a economy with a continuum of agents and obtained a characterization of insurable risks

– individual risks are insurable if and only if they are essentially pairwise independent.

In the private information economy with finite agents, the solution concepts are not equivalent. However, it is well-known that in a deterministic economy model, although solution concepts are defined from different perspectives, they may coincide with each other under certain assumptions. For instance, Aumann (1964) showed the equivalence between Walrasian equilibrium and core in a large deterministic game. Sun *et al.* (2013) examines the above-mentioned concepts (*i.e.*, Radner equilibrium, private core and insurance equilibrium) and shows that the same equivalence relationship continues to hold in the context of private information economy with a continuum of agents. Note that in the private information economy with a continuum of agents, besides the above-mentioned solution concepts, there are some others, *e.g.*, *ex ante* efficient core and *ex post* efficient core, and the equivalence may not still hold.

In the private information economy with finite agents, the solution concepts above-mentioned are automatically incentive compatible, and in the private information economy with a continuum of agents, the *ex ante* efficient core allocation is also incentive compatible; see Sun and Yannelis (2008a). In this paper, we will see that the private core allocation is not always incentive compatible (so are Radner equilibrium and insurance equilibrium). Compared with the private information economy with finite agents, this issue comes from the resource feasibility. In the private information economy with finite agents, feasibility is a restriction for the available allocation. However, in the private information economy with a continuum of agents, feasibility immediately follows the law of large numbers, and no allocation will be precluded by feasibility. To resolve this issue, by applying the maximin interim utility instead of expected interim utility, we will see every interim efficient private core allocation should be incentive compatible.

This chapter is organized as follows: in Section 5.2 we introduce a framework for the modeling of uncertainty and private information, a private information economy model and its induced large deterministic economy model; in Section 5.3 we define three equivalent solution concepts, Radner equilibrium, private core and insurance equilibrium; in Section 5.4, we show that the private core allocation is not always incentive compatible; in Section 5.5 we discuss related work in the literature; Section 5.6 contains all the proofs.

5.2 Modeling

5.2.1 Modeling of uncertainty and private information

We fix an atomless probability space $(I, \mathcal{I}, \lambda)$ representing the **space of agent**, and $S = \{s_1, s_2, \ldots, s_K\}$ the space of **macro states** of nature (its power set denoted by $\mathcal{S}$), which are not known to the agents. Let $T^0 = \{q_1, q_2, \ldots, q_L\}$ be the space of all the possible signals (types) for individual agents, and $(T, \mathcal{T})$ a measurable space that models the private signal profiles for all the agents, and therefore T is a space of functions from I to T^0.[1] Thus, $t \in T$, as a function from I to T^0, represents a private signal profile for all agents in I. For agent $i \in I$, $t(i)$ (also denoted by t_i) is the **private signal** of agent i while t_{-i} the restriction of the signal profile t to the set $I - \{i\}$ of agents different from i;[2] let T_{-i} be the set of all such t_{-i}. For simplicity, we shall assume that $(T, \mathcal{T})$ has a product structure so that T is a product of T_{-i} and T^0, while $\mathcal{T}$ is the product algebra of the power set $\mathcal{T}^0$ on T^0 with a σ-algebra $\mathcal{T}_{-i}$ on T_{-i}. For $t \in T$ and $t'_i \in T^0$, we shall adopt the usual notation (t_{-i}, t'_i) to denote the signal profile whose value is t'_i for agent i, and the same as t for other agents.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space representing all the uncertainty on the macro states as well as on the signals for all the agents, where $(\Omega, \mathcal{F})$ is the product measurable space $(S \times T, \mathcal{S} \otimes \mathcal{T})$. To avoid verbosity, we often refer to $\omega \in \Omega$ as a state which should nevertheless be distinguished from a macro state of nature. Let $\tilde{s}$ and $\tilde{t}_i, i \in I$ be the respective projection mappings from Ω to S and from Ω to T^0 with $\tilde{t}_i(s, t) = t_i$.[3] Let $\mathbf{P}_{t_i}^{S \times T_{-i}}$ be the conditional probability measure on the product space $(S \times T_{-i}, \mathcal{S} \otimes \mathcal{T}_{-i})$ when the signal of agent i is $t_i \in T^0$.

Let f be the **signal process** from $I \times \Omega$ to T^0 such that $f(i, \omega) = t_i$ for any $(i, t) \in I \times T$. Since Ω is the product of macro state space S and signal profile space T, $f(i, \omega)$ can also be written as $f(i, s, t)$ where $\omega = (s, t)$. We assume that f is $\mathcal{I} \boxtimes \mathcal{F}$-measurable.

[1] In the literature, one usually assumes that different agents have possibly different sets of signals and require that the agents take all their own signals with positive probability. For notational simplicity, we choose to work with a common set T^0 of signals, but allow zero probability for some of the redundant signals. There is no loss of generality in this latter approach.

[2] It is a notational convention in the literature that t_A refers to the restriction of a function t to the subset A and $t_{-A} = t_{A^c}$. When $A = \{i\}$ is a singleton, they will further be shortened to t_i and t_{-i} respetively.

[3] $\tilde{t}_i$ can also be viewed as a projection from T to T^0.

For each agent $i \in I$, we can define a probability distribution π_i of agent i's private information signal distribution. For $q \in T^0$, $\pi_i(\{q\}) = \mathbf{P} f_i^{-1}(\{q\})$ is the probability of agent i receiving a private information signal q. For notational simplicity, $\pi_i(\{q\})$ is often abbreviated as $\pi_i(q)$. Let $T_i = \{q \in T^0 \mid \pi_i(q) > 0\}$ be the set of all private information signals that matter to agent i (in the probabilistic sense). Since T^0 is finite, the agent space can be partitioned into a finite number of disjoint sets $I_1, \ldots, I_h$ such that all agents in $I_j, 1 \leq j \leq h$, have the same set of T_i. That is, $i_1, i_2 \in I_j$ for some $1 \leq j \leq h$ if and only if $T_{i_1} = T_{i_2}$. Thus, we can use $T_{I_j}, 1 \leq j \leq h$ to denote the common set of T_i for all agents $i \in I_j$. Fix $q \in T^0$, note that $\pi_i(q) = \int_\Omega 1_{f^{-1}(\{q\})}(i,\omega)\, \mathrm{d}\mathbf{P}$, hence $\pi_i(q)$ is a measurable function if viewed as a function on I. Since

$$I_j = \left(\cap_{q \in T_{I_j}} \{i \in I \mid \pi_i(q) > 0\}\right) \cap \left(\cap_{q \notin T_{I_j}} \{i \in I \mid \pi_i(q) = 0\}\right),$$

I_j is measurable for all $1 \leq j \leq h$. We assume that $\lambda(I_j) > 0$, $1 \leq j \leq h$.

5.2.2 Private information economy

We shall now follow the definition and notation in Section 5.2.1. We consider a large economy with asymmetric information. The space of agents is the atomless probability space $(I, \mathcal{I}, \lambda)$. In this economy, agents $i \in I$ are informed with their private signals $t_i \in T^0$ but not the macro state, and they can have contingent consumptions based on the signal profiles $t \in T$ announced by all the agents. Decisions are made at the *ex ante* level. The common consumption set is the positive orthant $\mathbb{R}^m_+$. In the sequel, we shall state several assumptions on the economy.

- The utility function of each agent depends on her/his consumption $z = (z_1, \ldots, z_m) \in \mathbb{R}^m_+$ and the private signal $q \in T^0$, where z_j is the quantity of the j-th commodity in the consumption z. Thus, we can let u be a function from $I \times \mathbb{R}^m_+ \times T^0$ to $\mathbb{R}_+$ such that for any given $i \in I$, $u(i, z, q)$ is the **utility** of agent i at consumption $z \in \mathbb{R}^m_+$ and the private signal $q \in T^0$.

- The utility function u is assumed to be measurable. For any given $q \in T^0$, $u(i, z, q)$ is continuous in $z \in \mathbb{R}^m_+$.

- Let e be an integrable function from $I \times T^0$ to $\mathbb{R}^m_+$ with $e(i, q)$ as the **initial endowment** of agent i.

We shall now consider an economy where the agents are informed with their signals but not the macro state. Formally, the collection

$$\mathcal{E} = \{(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P}), u, e, f, \tilde{s}\}$$

is called a **private information economy**.

5.2.3 Induced large deterministic economy

From the private information economy $\mathcal{E}$, we can construct an auxiliary large deterministic economy $\bar{\mathcal{E}}$; see also Sun (2006) and Malinvaud (1972). This large deterministic economy will be handy in the proofs of our results.

For each $i \in I$, we define a function E_i from $(\mathbb{R}^m_+)^{T_i}$ to $\mathbb{R}^m_+$ as

$$E_i(y) = \sum_{q \in T_i} \pi_i(q) \cdot y(q) = \int_{T^0} y(q) \, \mathrm{d}\pi_i.$$

Note that if y' is a random variable from T^0 to $\mathbb{R}^m_+$ and its restriction on T_i is y, then

$$\int_{T^0} y'(q) \, \mathrm{d}\pi_i(q) = \sum_{q \in T^0} \pi_i(q) \cdot y'(q) = E_i(y).$$

That means $E_i(y)$ is equal to the expectation of y'. For this reason, $E_i(y')$ will also be used to denote the expectation of y' on the probability measure space $(T^0, \mathcal{T}^0, \pi_i)$.

For any $z \in \mathbb{R}^m_+$, we define the following set

$$E_i^{-1}(z) = \left\{ y \in (\mathbb{R}^m_+)^{T_i} \mid E_i(y) = z \right\}.$$

The utility function v of the economy $\bar{\mathcal{E}}$ is defined as

$$v(i, z) = \max \left\{ \sum_{q \in T_i} \pi_i(q) \cdot u_i(y(q), q) \;\middle|\; y \in E_i^{-1}(z) \right\}$$

for agent i with consumption z.

The following lemma shows that v indeed defines a utility function.

Lemma 5.2.1. *For each $i \in I$, v_i is well-defined and continuous. Furthermore, if u_i is* *(strictly) monotone, then v_i is (strictly) monotone.*

This lemma is proved in Sun (2006) (see Lemma 3.2 therein) for the case of complete and continuous preference. An alternative proof that deals directly with utility function is also provided in Section 5.6.

Each agent $i \in I$ in the large deterministic economy $\bar{\mathcal{E}}$ has an initial endowment $\bar{e}(i) = E_i(e) = \int_\Omega e(i, f(i,\omega))\,\mathrm{d}\mathbf{P}$. We summarize the **induced large deterministic economy** as

$$\bar{\mathcal{E}} = \{(I, \mathcal{I}, \lambda), v, \bar{e}\}.$$

5.3 Equilibrium, core and insurance equilibrium

In this section, we introduce the definitions for Radner equilibrium, private core, insurance equilibrium, and related concepts.

A **price** is a measurable mapping from Ω to Δ_m, where Δ_m denotes the unit simplex in $\mathbb{R}^m_+$. For $\omega \in \Omega$, $p(\omega)$ is the commodity prices when ω occurs.

An **allocation** x is a measurable mapping from $I \times \Omega$ to $\mathbb{R}^m_+$. For any $(i,\omega) \in I \times \Omega$, $x(i,\omega)$ is interpreted as agent i's consumption at ω. When agents' consumption depends only on their private information signals, we simply write it as $x(i,q)$, where $q \in T^0$. In this case, x is also viewed as a mapping from $I \times T^0$ to $\mathbb{R}^m_+$.

Let y be a measurable mapping from Ω to $\mathbb{R}^m_+$. For agent $i \in I$, if her/his consumption plan is y, *i.e.*, her/his consumption is $y(\omega)$ when ω occurs, then her/his **(ex ante) expected utility** is given as

$$U_i(y) = \int_\Omega u_i(y(\omega), f(i,\omega))\,\mathrm{d}\mathbf{P}(\omega).$$

When agent i's consumption plan is contingent on her/his private information signal, *i.e.*, y is defined on T^0, her/his (ex ante) expected utility becomes

$$U_i(y) = \int_\Omega u_i\big(y(f(i,\omega)), f(i,\omega)\big)\,\mathrm{d}\mathbf{P}(\omega) = \sum_{q\in T^0} \pi_i(q) \cdot u_i(y(q), q) = E_i u_i(y(q), q).$$

One particular feature of Radner equilibrium is the possibility of wealth transfer across states. Hence, agents are no longer constrained to their income at each state. They may spend their income in advance or save it up for the future. The only requirement is that their expenditure remains balanced across the states. Consequently, they are faced with a new type of budget set. Given a price p, agent i's **interim budget set** is

$$B_i(p) = \left\{ y\colon \Omega \to \mathbb{R}^m_+ \;\middle|\; \int_\Omega p(\omega) \cdot y(\omega) \,\mathrm{d}\mathbf{P}(\omega) \leq \int_\Omega p(\omega) \cdot e(i, f(i,\omega)) \,\mathrm{d}\mathbf{P}(\omega) \right\}.$$

Now, we state the definition of Radner equilibrium:

Definition 5.3.1 (Radner Equilibrium or Walrasian Expectations Equilibrium)**.** *Let $\mathcal{E} = \{(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P}), u, e, f, \tilde{s}\}$ be a private information economy. A* ***Radner equilibrium*** *is a pair of an allocation x^* and a price p^* such that*

(1) For each agent $i \in I$, x^*_i depends on her private information signal only.

(2) x^* is feasible, i.e., $\int_I x^*(i, f(i,\omega)) \,\mathrm{d}\lambda = \int_I e(i, f(i,\omega)) \,\mathrm{d}\lambda$ for $\mathbf{P}$-almost all $\omega \in \Omega$.

(3) For λ-almost all agent $i \in I$, x^*_i is a maximizer of the following problem:

$$\begin{aligned} &\underset{y}{\text{maximize}} && U_i(y) \\ &\text{subject to} && y \in B_i(p^*). \end{aligned}$$

An allocation x for the private information economy $\mathcal{E}$ is called a ***Radner equilibrium allocation*** *if there is a price p such that (x, p) forms a Radner equilibrium. The set of all Radner equilibrium allocations is denoted by $RE(\mathcal{E})$.*

Condition (1) indicates that each agent's consumption is contingent on her/his private information signal. Condition (2) is the standard market clearing condition. Condition (3) shows that each agent makes an *ex ante* plan of consumption for all possible states the future may reveal to her/him and the plan maximizes her/his expected utility subject to interim budget set.

The concept of private core was initiated by Yannelis (1991). Before we formally define it, we need the following definition.

Definition 5.3.2. *Let x and x' be two allocations for the private information economy $\mathcal{E}$, W a coalition. The allocation x' is said to block the allocation x on W if*

(1) $\int_W x'(i,\omega) \,\mathrm{d}\lambda = \int_W e(i, f(i,\omega)) \,\mathrm{d}\lambda$ for $\mathbf{P}$-almost all $\omega \in \Omega$.

(2) $U_i(x'_i) > U_i(x_i)$ for λ-almost all $i \in W$.

When the allocation x' depends only on agents' private information signal, Condition (1) becomes:

$$\int_W x'(i, f(i,\omega))\, \mathrm{d}\lambda = \int_W e(i, f(i,\omega))\, \mathrm{d}\lambda \text{ for } \mathbf{P}\text{-almost all } \omega \in \Omega.$$

If u is continuous and strictly monotone, Condition (2) is equivalent to:

$$U_i(x'_i) \geq U_i(x_i) \text{ for } \lambda\text{-almost all } i \in W \text{ and } \lambda(\{i \in W \mid U_i(x'_i) > U_i(x_i)\}) > 0.$$

We give the definition of private core below:

Definition 5.3.3 (Private Core)**.** *Let $\mathcal{E} = \{(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P}), u, e, f, \tilde{s}\}$ be a private information economy. The **private core** of $\mathcal{E}$ is the set of all allocations x such that*

(1) For all $i \in I$, x_i depends only on agent i's private information signal.

(2) x is feasible, i.e., $\int_I x(i, f(i,\omega))\, \mathrm{d}\lambda = \int_I e(i, f(i,\omega))\, \mathrm{d}\lambda$ for $\mathbf{P}$-almost all $\omega \in \Omega$.

(3) There is no coalition W and no allocation x' which depends only on private information signal such that x' blocks x on W.

We denote the private core by $PC(\mathcal{E})$.

In the definition of private core, we notice that there is no information sharing among agents in a coalition. Each agent uses only their own private information signal in making consumption plan. For the philosophy behind this definition, we quote Yannelis (1991) "... since in most applications, agents do not have an incentive to reveal their own private information (think of situations of moral hazard or adverse selection)."

The third solution concept we will encounter is called insurance equilibrium. When there are a large number of risk bearing agents in a market and no collective risk prevails, it is often conjectured that contingent commodity prices are the multiple of "sure prices" and an objective probability; see Malinvaud (1972, 1973). To justify it, as shown in Hildenbrand (1971), one can alter the standard state-wise market clearing condition for economy with uncertainty by equating the aggregate expected demand to the aggregate expected supply (in the case when there is no production, the aggregate supply is equal to the aggregate initial endowments). The exact law of large numbers (which requires a large number of agents and some kind of independence between them) indicates that market clearing condition is also satisfied state-wise. Since this model is often used for the study of insurance system (see Malinvaud (1972) and Sun (2006)), we call the corresponding equilibrium concept insurance equilibrium. It should be reminded, however, that there is no standard name for this equilibrium in the literature. We define insurance equilibrium below:

Definition 5.3.4 (Insurance Equilibrium)**.** *Let* $\mathcal{E} = \{(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbf{P}), u, e, f, \tilde{s}\}$ *be a private information economy. An* ***insurance equilibrium*** *is a pair of an allocation* x^* *and a price* p^* *such that*

(1) The price p^* *is constant, i.e.,* $p^*(\omega) = p^*(\omega')$ *for all* $\omega, \omega' \in \Omega$.

(2) For each agent $i \in I$, x_i^* *depends only on her/his private information signal.*

(3) The expected allocation $E_i(x_i^*)$ *is feasible in the sense that*

$$\int_I E_i(x^*(i, f(i,\omega)))\,\mathrm{d}\lambda = \int_I E_i(e(i, f(i,\omega)))\,\mathrm{d}\lambda.$$

(4) For λ*-almost all agent* $i \in I$, x_i^* *is a maximizer of the following problem:*

$$\begin{aligned} &\underset{y}{\textit{maximize}} && U_i(y) \\ &\textit{subject to} && y \in B_i(p^*). \end{aligned}$$

An allocation x *for the private information economy* $\mathcal{E}$ *is called an* ***insurance equilibrium allocation*** *if there is a price* p *such that* (x, p) *forms an insurance equilibrium. The set of all insurance equilibrium allocations is denoted by* $IE(\mathcal{E})$.

Condition (1) shows that the price in equilibrium is a "sure price" in that it does not depend on ω. Condition (2) indicates that each agent's consumption is contingent on her/his private information signal. Condition (3) is the market clearing condition for expected demand and expected supply. This condition is special since market clearing condition is usually required to hold state-wise (*i.e.*, for each $\omega \in \Omega$). As we mentioned above, under certain assumptions the law of large numbers implies that market is cleared for each state ω when Condition (3) holds. As it is evident that state-wise market clearing condition implies Condition (3), in fact these two conditions are equivalent provided that the law of large numbers holds. Condition (4) requires that each agent act to maximize their expected utility subject to the constraint of interim budget set. When price is constant, we notice that the interim budget set becomes

$$B_i(p) = \left\{ y \colon \Omega \to \mathbb{R}^m_+ \;\middle|\; p \cdot \int_\Omega y(\omega)\,\mathrm{d}\mathbf{P} \le p \cdot \int_\Omega e(i, f(i,\omega))\,\mathrm{d}\mathbf{P} \right\}.$$

Hence agent i chooses a contingent plan of consumption whose expected value does not exceed the value of expected initial endowment.

By now, we have defined three solution concepts for the private information economy. We state the equivalence result in the following, which has been showed in Sun *et al.* (2013).

We make the following assumptions firstly.

A1 For **P**-almost all $\omega \in \Omega$, $\int_I e(i, f(i,\omega))\,\mathrm{d}\lambda \gg 0$.

A2 For **P**-almost all $\omega \in \Omega$, $\int_I e(i, f(i,\omega))\,\mathrm{d}\lambda \ll \infty$.

A3 For any fixed $i \in I$, $q \in T^0$, the utility function $u(i, \cdot, q)$ is continuous and strictly monotone.

A4 For any fixed $i \in I$, $q \in T^0$, the utility function $u(i, \cdot, q)$ is concave.

A5 The signal process f is essentially pairwise independent in the sense that for λ-almost all $i \in I$, f_i and f_j are independent for λ-almost all $j \in I$.

The first four assumptions are standard in equilibrium analysis and need no explanation. The last assumption indicates that for any two individuals, their private information signals are pairwise independent. As we shall see later, that under this assumption, each individual agent is informationally negligible. A detailed discussion on this topic can be found in Sun (2006).

The following result is proved by Sun *et al.* (2013).

Fact 5.3.5. *Let $\mathcal{E}$ be a private information economy. Then Radner equilibrium, insurance equilibrium and private core coincide in $\mathcal{E}$ in the sense that $IE(\mathcal{E}) = RE(\mathcal{E}) = PC(\mathcal{E})$.*

5.4 Incentive compatibility

Koutsougeras and Yannelis (1993) show the incentive compatibility of the private core allocation. In this section, we will consider the incentive compatibility for the three solution concepts defined in Section 5.3, and will find that they are always not incentive compatible. Since the set of macro states are finite, without loss of generality, we take it to be a singleton set for sake of simplicity. When S is a singleton set, $\mathbf{P}$ and $\mathbf{P}^T$ are identical, and so are $\mathbf{P}_{t_i}^{T_{-i}}$ and $\mathbf{P}_{t_i}^{S\times T_{-i}}$.

Definition 5.4.1. *For an allocation x, an agent $i \in I$, private signals $t_i, t_i' \in T^0$, let*

$$U_i(x_i, t_i' \mid t_i) = \int_{T_{-i}} u_i\Big(e_i(t_i) + x_i(t_{-i}, t_i') - e_i(t_i'), t_i\Big)\, \mathrm{d}\mathbf{P}_{t_i}^{T_{-i}},$$

be the interim expected utility of agent i when she receives private signal t_i but mis-report as t_i'. The allocation x is said to be incentive compatible if for λ-almost all $i \in I$,

$$U_i(x_i, t_i \mid t_i) \geq U_i(x_i, t_i' \mid t_i)$$

holds for all the non-redundant signals $t_i, t_i' \in T^0$ of agent i (i.e., $\pi_i(t_i), \pi_i(t_i') > 0$).

The following proposition shows that the private core allocation is not always incentive compatible, where agent's endowment does not depend on her/his private signal.

Proposition 5.4.2. *There exists a large private information economics $\mathcal{E} = \{(I\times\Omega, \mathcal{I}\boxtimes\mathcal{F}, \lambda\boxtimes\mathbf{P}), u, e, f, \tilde{s}\}$, such that every private core allocation is not incentive compatible,*[4] *where agent i's endowment is a constant function, and agent i's utility is a function from $\mathbb{R}_+^m \times T^0$ to $\mathbb{R}_+$.*

We also show that the private core allocation is not always incentive compatible, where agent's endowment depends on her/his private signal but agent's utility only depends on the allocation.

Proposition 5.4.3. *There exists a large private information economics $\mathcal{E} = \{(I\times\Omega, \mathcal{I}\boxtimes\mathcal{F}, \lambda\boxtimes\mathbf{P}), u, e, f, \tilde{s}\}$, such that every private core allocation is not incentive compatible, where agent i's endowment is a function from T^0 to $\mathbb{R}_+^m$, and agent i's utility is a function from $\mathbb{R}_+^m$ to $\mathbb{R}_+$.*

[4] In this position and the following one, the private core is not required to satisfy Condition (1).

5.5 Discussion

The study of the equivalence between core and competitive equilibrium first appeared in Edgeworth (1881). In the book Edgeworth showed, in a very special setup, that core collapsed to the set of competitive equilibria as the number of agents in an economy gets large. He continued to conjecture that this equivalence relationship should hold for a general economy. Edgeworth's conjecture was first proved by Debreu and Scarf (1963). Anderson (1978) proved a Core Equivalence Theorem with the help of Shapley-Folkman Theorem. Following a argument similar to Anderson's, Aumann (1964) obtained the same result for large economies using Lyapunov Theorem. Anderson (1992) is a good reference for a comprehensive survey on core equivalence theorems.

In this chapter, we investigate the relationship among private core, Radner equilibrium and insurance equilibrium in the private information economy model. We show that these three concepts coincide in a large economy with private information provided that agents' private information signals are essentially pairwise independent.

The work in the literature that is closely related to ours is Einy *et al.* (2001). Our work differs from theirs mainly in two aspects. In their paper, Einy, Moreno and Shitovitz establish the existence of Radner equilibrium for "irreducible" large economy with private information. An economy is "irreducible" if a coalition can always improve its welfare with another coalition's initial endowments. Einy et. al. further show that Radner equilibrium and private core coincide in an "irreducible" economy. While in our work, we do not impose the "irreducibility" assumption on the private information economy. Furthermore, in their paper private information is modeled by partitions of the macro state space for each agent. On the other hand, we use a signal process and private information signals to model private information. This allows us to consider the informational negligibility of an individual agent. When the signal process is essentially pairwise independent, the exact law of large numbers indicates that each agent has negligible information. However, this is not so clear with their model although they also consider an economy with a continuum of agents.

Sun and Yannelis (2007a) have also proved the core equivalence theorem for a large private information economy. However, it is worth pointing out that the equilibrium and core in their paper are defined in an *ex ante* sense. In particular, equilibrium allocation and core depend on the aggregate signals. Private information plays its role in the study of incentive compatibility. On the other hand, in our definitions an allocation depends only on agents' private information signals. Hence, the concepts of equilibrium and core in these two work are different.

5.6 Proofs of Propositions 5.4.2 and 5.4.3

We take S to be a singleton set. Then we can identify $(\Omega, \mathcal{F}, \mathbf{P})$ with $(T, \mathcal{T}, \mathbf{P}^T)$, where $\mathbf{P}^T$ is the marginal probability measure of $\mathbf{P}$ on $(T, \mathcal{T})$. The construction will use nonstandard analysis. One can pick up some background knowledge on nonstandard analysis from the first three chapters of the book Loeb and Wolff (2000).

Fix $n \in {}^*\mathbb{N}_\infty$. Let $I = \{1, 2, \ldots, n\}$ with internal power set $\mathcal{I}_0$ and internal counting probability measure λ_0 on $\mathcal{I}_0$ with $\lambda_0(A) = |A|/|I|$ for any $A \in \mathcal{I}_0$, where $|A|$ is the internal cardinality of A. Let $(I, \mathcal{I}, \lambda)$ be the Loeb space of the internal probability space $(I, \mathcal{I}_0, \lambda_0)$, which will serve as the space of agents for the large private information economy considered below.

Let $T^0 = \{0, 1\}$ be the signals for individual agents, and T the set of all the internal functions from I to T^0 (the space of signal profiles). Let $\mathcal{T}_0$ be the internal power set on T, $\mathbf{P}_0$ an internal counting probability measure on $(T, \mathcal{T}_0)$ (*i.e.*, the probability weight for each $t = (t_1, t_2, \ldots, t_n) \in T$ under $\mathbf{P}_0$ is $1/2^n$), and $(T, \mathcal{T}, \mathbf{P})$ the corresponding Loeb space.

Let $(I \times T, \mathcal{I}_0 \otimes \mathcal{T}_0, \lambda_0 \otimes \mathbf{P}_0)$ be the internal product probability space of $(I, \mathcal{I}_0, \lambda_0)$ and $(T, \mathcal{T}_0, \mathbf{P}_0)$. Let $(I \times T, \mathcal{I} \boxtimes \mathcal{T}, \lambda \boxtimes \mathbf{P})$ be the Loeb space of the internal product $(I \times T, \mathcal{I}_0 \otimes \mathcal{T}_0, \lambda_0 \otimes \mathbf{P}_0)$, which is indeed a Fubini extension of the usual product probability space by Keislers Fubini Theorem (see, for example Section 5.3.7 in Loeb and Wolff (2000)).

Proof of Proposition 5.4.2. We consider a one-good economy with utility functions $u(i, z, q) = (1 + q)\sqrt{z}$ and constant endowments $e(i, q) = 1$ for all the agents $i \in I$ and $q \in T^0$.

Let x be a private core allocation, then for all $i \in I$, x_i depends only on agent i's private information signal t_i. Let $a_i = \int_T x_i(t)\, \mathrm{d}\mathbf{P} = \frac{1}{2}x_i(0) + \frac{1}{2}x_i(1)$, then Jensen's inequality implies that

$$U_i(x_i) = \sum_{t_i \in T^0} \pi_i(t_i) u(i, x(i, t_i), t_i) = \frac{1}{2} u_i(x_i(0), 0) + \frac{1}{2} u_i(x_i(1), 1)$$
$$= \frac{1}{2}\sqrt{x_i(0)} + \sqrt{x_i(1)} \leq \sqrt{\frac{5a_i}{2}}$$

with equality only when $x_i(0) = \frac{2}{5}a_i$ and $x_i(1) = \frac{8}{5}a_i$.

Define an allocation y by letting $y(i,t) = \frac{2}{5}(1+3t_i)a_i$. By exact law of large numbers, we have

$$\int_I y(i,t)\,\mathrm{d}\lambda = \int_I \int_T \frac{2}{5}(1+3t_i)a_i\,\mathrm{d}\mathbf{P}\,\mathrm{d}\lambda = \int_I \frac{2}{5}(1+3/2)a_i\,\mathrm{d}\lambda = \int_I a_i\,\mathrm{d}\lambda = 1,$$

for $\mathbf{P}$-almost all $t \in T$. That is, the allocation y satisfies the first two conditions in Definition 5.3.3.

On the other hand, we have

$$U_i(y_i) = \sqrt{\frac{5a_i}{2}}.$$

Hence $U_i(y_i) \geq U_i(x_i)$ with equality only when $x_i(0) = 2a_i/5$ and $x_i(1) = 8a_i/5$.

Since x is a private core allocation, it is *ex ante* efficient, and hence there exists a set A in I with $\lambda(A) = 1$ such that for any $i \in A$, $x_i(0) = 2a_i/5$ and $x_i(1) = 8a_i/5$. Let $B = \{i \in A \mid a_i > 0\}$. Since $\int_I a_i\,\mathrm{d}\lambda = 1$, we have $\lambda(B) > 0$. By the definition of incentive compatibility, we have, for any agent $i \in B$,

$$U_i(x_i, 1 \mid 0) = u_i(x_i(1), 0) = \sqrt{8a_i/5} > \sqrt{2a_i/5} = u_i(x_i(0), 0) = U_i(x_i, 0 \mid 0).$$

Hence, x is not incentive compatible. □

Proof of Proposition 5.4.3. We consider a one-good economy with strictly concave and monotonic utility functions $u_i\colon \mathbb{R}_+ \to \mathbb{R}_+$ and endowments $e_i(t_i) = 2t_i$ for all $i \in I$. By the exact law of large numbers, we have $\int_I e_i(t_i)\,\mathrm{d}\lambda = \int_I \int_T 2t_i\,\mathrm{d}\mathbf{P}\,\mathrm{d}\lambda = 1$ for $\mathbf{P}$-almost all $t \in T$.

Let x be a private core allocation. As in the proof of Theorem 1 in Sun and Yannelis (2008a), we will have $x_i(t) = \int_T x_i(t)\,\mathrm{d}\mathbf{P}(t)$ for $\mathbf{P}$-almost all $t \in T$.

By the definition of incentive compatibility, we have, for λ-almost all agents $i \in B$,

$$\begin{aligned} U_i(x_i, 0 \mid 1) &= \int_{T_{-i}} u_i\big(e_i(1) - e_i(0) + x_i(t_{-i}, 0)\big) \, \mathrm{d}\mathbf{P}^{T_{-i}} \\ &= \int_{T_{-i}} u_i(x_i(t_{-i}, 1) + 2) \, \mathrm{d}\mathbf{P}^{T_{-i}} \\ &> \int_{T_{-i}} u_i(x_i(t_{-i}, 1)) \, \mathrm{d}\mathbf{P}^{T_{-i}} = U_i(x_i, 1 \mid 1). \end{aligned}$$

Hence, x is not incentive compatible. □

Chapter 6

Concluding remarks

The chapter concludes its comprehensive overview of the results that have been obtained by suggesting several research topics, some of which remain open.

6.1 Random matching

In the static case, the foundations of the independent random (full and partial) matching with a continuum population and general types have been established by Duffie and Sun (2007, 2012) and Sun (2013a).

In the discrete-time dynamic case, Duffie and Sun (2007, 2012) provide the micro foundation for the independent random full matching with general types and the independent random partial matching with finite types, and Sun (2013b) provides the foundation for the independent random partial matching with general types.

In the continuous-time dynamic case, Duffie *et al.* (2013a) show the existence of independent random matching of a large population and finite types. In particular, they construct a continuum of independent continuous-time Markov processes that is derived from random mutation, random partial matching and random type changing. The empirical type evolution of such a continuous-time dynamic system is also determined.

Besides, Duffie *et al.* (2013b) provide micro foundation for independent random matching with directed probabilities, where the matching probabilities are type-relevant and exogenous.

6.2 Game theory

In He, Sun and Sun (2013), we propose the condition of "nowhere equivalence" to model the space of agents. We show that this condition is more general than all of those special conditions imposed on the spaces of agents to handle the failure of the classical Lebesgue unit interval. We also illustrate the optimality of this condition by showing its necessity in deriving the existence of pure-strategy Nash equilibrium for nonatomic games. Actually, the results in Chapter 4 can be implied by the results in He, Sun and Sun (2013).

Furthermore, He and Sun (2013) study the existence of pure-strategy equilibria for the finite-player game with incomplete information based on the condition of "nowhere equivalent". We show that the condition of "nowhere equivalent" to model the information space is a necessary and sufficient condition to guarantee the existence of pure-strategy equilibria.

6.3 General equilibrium

The theory of the value, as shaped by Aumann and Shapley (1974). In Aumann (1975), where the set of agents by Lebesgue unit interval or some probability isomorphic, Aumann (1975) showed the equivalence between the value allocation and the competitive allocation in a model of large economy. In Aumann (1975)'s paper, he implicitly used the idea of nonstandard analysis in the sense that each player in the large economy occupies ϵ weight in the economy, where ϵ represents an infinitesimal number. In a follow up paper, Brown and Loeb (1976) re-establish the equivalence between value and competitive allocations, while the set of agents is modeled by a hyperfinite counting set, or a hyperfinite Loeb counting space. Now, as an appeal of a general technique of Keisler and Sun (2009), it is of interest to investigate whether such a value equivalence result still holds when modeling the set of agents by saturated probability spaces, and above all, it still remains open that how to define the value in such a large economy.

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